

# Instantons on sine-cones over Sasakian manifolds

Severin Bunk<sup>†</sup>, Tatiana A. Ivanova\*, Olaf Lechtenfeld<sup>†×</sup>,  
Alexander D. Popov<sup>†</sup> and Marcus Sperling<sup>†</sup>

<sup>†</sup>*Institut für Theoretische Physik  
Leibniz Universität Hannover  
Appelstraße 2, 30167 Hannover, Germany*

Email: Severin.Bunk, Olaf.Lechtenfeld, Alexander.Popov,  
Marcus.Sperling@itp.uni-hannover.de

<sup>×</sup>*Riemann Center for Geometry and Physics  
Leibniz Universität Hannover  
Welfengarten 1, 30167 Hannover, Germany*

<sup>\*</sup>*Bogoliubov Laboratory of Theoretical Physics, JINR  
141980 Dubna, Moscow Region, Russia*

Email: ita@theor.jinr.ru

## Abstract

We investigate instantons on sine-cones over Sasaki-Einstein and 3-Sasakian manifolds. It is shown that these conical Einstein manifolds are Kähler with torsion (KT) manifolds admitting Hermitian connections with totally antisymmetric torsion. Furthermore, a deformation of the metric on the sine-cone over 3-Sasakian manifolds allows one to introduce a hyper-Kähler with torsion (HKT) structure. In the large-volume limit these KT and HKT spaces become Calabi-Yau and hyper-Kähler conifolds, respectively. We construct gauge connections on complex vector bundles over conical KT and HKT manifolds which solve the instanton equations for Yang-Mills fields in higher dimensions.

# 1 Introduction

Kähler geometry is by now an established mathematical field with strong interrelations to theoretical physics, especially to supersymmetry and string theory. Another important geometry used in string theories and M-theory is Sasakian geometry (see [1]-[3] for reviews and references), especially in the AdS/CFT correspondence in the physically interesting dimensions five and seven. Here, we shall focus on Sasaki-Einstein manifolds and on 3-Sasakian manifolds, which are of dimension  $2n+1$  and  $4n+3$ , respectively.

On a manifold  $M$  of real dimension  $2n+1$ , Sasakian geometry is sandwiched between Kählerian geometries on particular manifolds of the two neighboring dimensions. In particular, the metric cone over any Sasakian manifold is Kähler, and over any 3-Sasakian manifold it is hyper-Kähler. If the metric on a Sasakian manifold is Einstein, then the metric cone over it is a Calabi-Yau manifold. As examples, some well-known homogeneous and some recently discovered inhomogeneous Sasakian spaces often occur in string compactifications [4]-[6]. These manifolds admit a connection with non-vanishing torsion and a structure group of  $SU(n) \subset SO(2n+1)$  or  $Sp(n) \subset SO(4n+3)$ , respectively. There exist brane solutions of ten-dimensional supergravity which interpolate between an  $AdS_{p+1} \times X_{9-p}$  near-horizon geometry and an asymptotic geometry  $M_p \times C(X_{9-p})$ , where  $M_p$  is  $p$ -dimensional Minkowski space and  $C(X_{9-p})$  is a metric cone over  $X_{9-p}$  (see e.g. [7, 8] and references therein). Such kinds of brane solutions in heterotic supergravity with Yang-Mills instantons on the metric cones  $C(X_{9-p})$  were considered in [9, 10]. We intend to generalize them by considering sine-cones with a Kähler-torsion structure instead of metric cones with a Kähler structure.

In this paper we will consider sine-cones over Sasaki-Einstein and 3-Sasakian manifolds and solve instanton equations for Yang-Mills fields on these conical manifolds. Recall that for any Riemannian metric  $g$  on a manifold  $M$ , the warped product metric  $\tilde{g}$  on  $C(M) = \mathbb{R}_+ \times M$  is defined as

$$\tilde{g} = dr^2 + f^2(r)g, \tag{1.1}$$

where  $r \in \mathbb{R}_+$  and  $f(r)$  is a *warping* function.  $(C(M), \tilde{g})$  is called the *metric cone* over  $M$  if  $f(r) = r$ , and it is known as the *sine-cone* over  $M$  if  $f(r) = \sin r$ . It is known that the sine-cone metric  $\tilde{g}$  over a Sasaki-Einstein or 3-Sasakian manifold is Einstein [2].<sup>1</sup> We show that these conical manifolds admit Kähler with torsion (KT) structures, i.e. on them there exists a Hermitian connection with a totally antisymmetric torsion  $\tilde{T} = Jd\tilde{\omega}$ . Here,  $J$  is an almost complex structure, and  $\tilde{\omega}(\cdot, \cdot) = \tilde{g}(J\cdot, \cdot)$  is the fundamental 2-form. For  $\tilde{T} = 0$  the Hermitian structure is Kähler.

We also show that for any 3-Sasakian manifold one can deform the sine-cone metric  $\tilde{g}$  in such a way that  $C(M)$  will be a hyper-Kähler with torsion (HKT) manifold. HKT geometry has been described in detail e.g. in [11] and intensively studied since then (see e.g. [12]-[15] and references therein). In fact, these Hermitian manifolds with three integrable almost complex structures  $J^\alpha$ ,  $\alpha = 1, 2, 3$ , are not hyper-Kähler for non-vanishing torsion  $\tilde{T} = J^1d\tilde{\omega}^1 = J^2d\tilde{\omega}^2 = J^3d\tilde{\omega}^3$ , but we will follow the established terminology. Here,  $\tilde{\omega}^\alpha(\cdot, \cdot) = \tilde{g}(J^\alpha \cdot, \cdot)$  are three Hermitian 2-forms.

After introducing the KT and HKT structures on the sine-cones over Sasaki-Einstein and 3-Sasakian manifolds, we consider the appropriate instanton equations on these conical manifolds. Such first-order gauge equations, which generalize the Yang-Mills anti-self-duality equations in  $d=4$  to higher-dimensional manifolds with special holonomy (or, more generally,  $G$ -structure), were introduced and studied both in the physical [16]-[18] and mathematical [19]-[21] literature. Some instanton solutions were found e.g. in [22]-[28]. Due to the foliated structure of the considered conical KT and HKT manifolds, a natural ansatz for the gauge fields reduces the instanton equations to matrix-model equations. We discuss their simplest analytic and numerical solutions.

---

<sup>1</sup>The corresponding metric-cone metric is even Ricci-flat.

This article is arranged as follows. In Section 2 we collect various geometric facts concerning Sasaki-Einstein, 3-Sasakian, Kähler-torsion and hyper-Kähler with torsion manifolds. We introduce the KT and HKT structures on sine-cones over Sasaki-Einstein and 3-Sasakian manifolds. In Section 3 we discuss the instanton equations in more than four dimensions and specialize them for KT and HKT manifolds. Then we describe an ansatz reducing these instanton equations to matrix equations and present some solutions.

## 2 Hermitian manifolds with torsion

**KT manifolds.** A  $G$ -structure on a smooth orientable manifold  $M$  of dimension  $m$  is a reduction of the structure group  $\mathrm{GL}(m, \mathbb{R})$  of the tangent bundle  $TM$  to a closed subgroup  $G \subset \mathrm{GL}(m, \mathbb{R})$ . Choosing an orientation and a Riemannian metric  $g$  defines an  $\mathrm{SO}(m)$ -structure on  $M$ . We assume that  $m = 2n$  is even and that  $(M, g)$  is an almost Hermitian manifold. This means that there exists an almost complex structure  $J \in \mathrm{End}(TM)$ , with  $J^2 = -\mathbf{1}_{TM}$ , which is compatible with the metric  $g$ , i.e.  $g(JX, JY) = g(X, Y)$  for all  $X, Y \in TM$ . One can introduce the fundamental two-form  $\omega$  as

$$\omega(X, Y) := g(JX, Y) \quad \text{for } X, Y \in TM, \quad (2.1)$$

and the canonical objects  $(g, J, \omega)$  define a  $U(n)$ -structure on  $M$ . The additional existence of a complex  $n$ -form  $\Omega$  reduces the  $U(n)$ - to an  $SU(n)$ -structure. It implies that the almost Hermitian manifold  $(M, g, J)$  has a topologically trivial canonical bundle. If the almost complex structure  $J$  is integrable, then  $(M, g, J)$  is a Hermitian manifold and  $(M, g, J, \Omega)$  is Calabi-Yau.

On any almost Hermitian manifold  $(M, g, J)$  one has a Hermitian connection<sup>2</sup> with totally antisymmetric torsion  $T$ . Two particularly interesting cases arise when the torsion  $T$  is the real part of either a  $(3,0)$ -form or a  $(2,1)$ -form. In the former case, the manifold  $M$  is called nearly Kähler [29, 30]. The latter case entails a property called Kähler-torsion (KT) [31, 32]. Namely, on a Hermitian manifold  $(M, g, J)$  the KT connection is a Hermitian connection  $\Gamma$  with antisymmetric torsion  $T$  given by

$$T = Jd\omega, \quad (2.2)$$

where  $J$  acts on the  $p$ -form  $e^{a_1} \wedge \cdots \wedge e^{a_p}$  as

$$J(e^{a_1} \wedge e^{a_2} \wedge \cdots \wedge e^{a_p}) = J e^{a_1} \wedge J e^{a_2} \wedge \cdots \wedge J e^{a_p} \quad \text{with } J e^a = J_b^a e^b. \quad (2.3)$$

Here  $\{e^a\}$  with  $a = 1, \dots, 2n$  is a local frame for the cotangent bundle  $T^*M$ , and  $J_b^a$  are the corresponding components of the almost complex structure  $J$ . Note that  $J$  is integrable in the KT case and non-integrable in the nearly Kähler case. For traceless anti-Hermitian  $\Gamma$  the KT manifolds are called Calabi-Yau torsion [33].

**Remark.** Let

$$\theta^j := e^{2j-1} + i e^{2j} \quad \text{and} \quad \theta^{\bar{j}} := \overline{\theta^j} \quad \text{with } j = 1, \dots, n \quad (2.4)$$

constitute a local frame for the  $(1,0)$  and  $(0,1)$  parts of the complexified cotangent bundle. Then  $J, g$  and  $\omega$  may be chosen as follows:

$$J\theta^j = i\theta^j, \quad g = \sum_{j=1}^n \theta^j \otimes \theta^{\bar{j}} \quad \text{and} \quad \omega = \frac{i}{2} \sum_{j=1}^n \theta^j \wedge \theta^{\bar{j}}. \quad (2.5)$$

Their non-vanishing components are thus given by

$$J_k^j = i\delta_k^j, \quad J_{\bar{k}}^{\bar{j}} = -i\delta_{\bar{k}}^{\bar{j}}, \quad g_{j\bar{k}} = \frac{1}{2}\delta_{j\bar{k}} \quad \text{and} \quad \omega_{j\bar{k}} = \frac{i}{2}\delta_{j\bar{k}} \quad (2.6)$$

---

<sup>2</sup>This connection  $\nabla = d + \Gamma$  preserves  $g, J$  and  $\omega$ .

with respect to the  $\theta$ -basis. For  $g^{-1}$  and  $\omega^{-1}$  we have

$$g^{j\bar{k}} = 2\delta^{j\bar{k}} \quad \text{and} \quad \omega^{j\bar{k}} = -2i\delta^{j\bar{k}} . \quad (2.7)$$

**Cones.** For any Riemannian manifold  $(\mathcal{M}, g)$  we define

$$C_\Lambda(\mathcal{M}) = ((0, \Lambda\pi) \times \mathcal{M}, \tilde{g}) \quad \text{with} \quad \tilde{g} = dr^2 + \Lambda^2 \sin^2\left(\frac{r}{\Lambda}\right) g \quad (2.8)$$

to be the *sine-cone* over  $\mathcal{M}$ . Here  $r \in (0, \Lambda\pi)$  which is an open interval, and so the volume of the sine-cone is  $\Lambda \text{vol}\mathcal{M}$ . In the infinite-volume limit  $\Lambda \rightarrow \infty$  the sine-cone becomes the *metric cone*<sup>3</sup>

$$C_\infty(\mathcal{M}) \equiv C(\mathcal{M}) = (\mathbb{R}_+ \times \mathcal{M}, \tilde{g}) \quad \text{with} \quad \tilde{g} = dr^2 + r^2 g . \quad (2.9)$$

Note that

$$\tilde{g} = dr^2 + \Lambda^2 \sin^2\left(\frac{r}{\Lambda}\right) g =: \Lambda^2 \sin^2\varphi (d\tau^2 + g) , \quad (2.10)$$

where

$$\varphi = \frac{r}{\Lambda} \quad \text{and} \quad \tau = \log\left(2\frac{\Lambda}{r_0} \tan\frac{\varphi}{2}\right) \Leftrightarrow r = 2\Lambda \arctan\left(\frac{r_0 e^\tau}{2\Lambda}\right) \quad (2.11)$$

with  $\tau \in \mathbb{R}$  and a constant  $r_0 \in \mathbb{R}_+$ . In the limit  $\Lambda \rightarrow \infty$ , (2.11) simplifies to

$$\tau = \log\left(\frac{r}{r_0}\right) \Leftrightarrow r = r_0 e^\tau \quad (2.12)$$

which are valid for the metric cone with

$$\tilde{g} = dr^2 + r^2 g = r^2 \left(\frac{dr^2}{r^2} + g\right) = r_0^2 e^{2\tau} (d\tau^2 + g) . \quad (2.13)$$

It follows from (2.12) and (2.13) that both cones are conformally equivalent to the cylinder

$$\text{Cyl}(\mathcal{M}) = (\mathbb{R} \times \mathcal{M}, g_{\text{cyl}}) \quad \text{with} \quad g_{\text{cyl}} = d\tau^2 + g . \quad (2.14)$$

**Sasaki-Einstein manifolds.** A  $(2n+1)$ -dimensional Riemannian manifold  $(\mathcal{M}, g)$  is called *Sasakian* if the metric cone  $(C(\mathcal{M}), \tilde{g})$  is Kähler.<sup>4</sup> A Sasakian manifold  $(\mathcal{M}, g)$  is *Sasaki-Einstein* if in addition the metric  $g$  is Einstein. In this case the metric cone  $C(\mathcal{M})$  is a Ricci-flat Kähler manifold (Calabi-Yau), so its holonomy group is reduced from  $U(n)$  to  $SU(n)$ . Sasaki-Einstein manifolds have a reduced structure group of  $SU(n) \subset SO(2n+1)$ . They are endowed with 1-, 2-, 3- and 4-forms  $\eta, \omega, P$  and  $Q$ , which can be defined in a local orthonormal basis  $\{e^{\hat{a}}\}$ ,  $\hat{a} = 1, \dots, 2n+1$ , as

$$\eta = -e^{2n+1} , \quad \omega = \sum_{j=1}^n e^{2j-1} \wedge e^{2j} , \quad P = \eta \wedge \omega \quad \text{and} \quad Q = \frac{1}{2} \omega \wedge \omega . \quad (2.15)$$

These forms satisfy the relations

$$de^{2n+1} = -2\omega \quad \text{and} \quad dP = 4Q . \quad (2.16)$$

Note that, in the above basis, the torsion  $T$  of the canonical  $su(n)$ -valued connection  $\Gamma$  on  $\mathcal{M}$  is not totally antisymmetric and has the components (see e.g. [9])

$$T^a = \frac{n+1}{2n} P_{\hat{a}\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}} \quad \text{and} \quad T^{2n+1} = P_{2n+1\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}} , \quad (2.17)$$

where  $\{\hat{a}\} = \{a, 2n+1\}$  with  $a = 1, \dots, 2n$ .

<sup>3</sup>One usually omits the adjective “metric” and simply says “cone”.

<sup>4</sup>This is one of several equivalent definitions of Sasakian manifolds [2].

**3-Sasakian manifolds.** A  $(4n+3)$ -dimensional Riemannian manifold  $(\mathcal{M}, g)$  is called *3-Sasakian* if the metric cone  $(C(\mathcal{M}), \tilde{g})$  on  $\mathcal{M}$  is hyper-Kähler. The structure group of  $\mathcal{M}$  then is  $\text{Sp}(n) \subset \text{SO}(4n+3)$ , and we let the index  $a$  run from 1 to  $4n$ . Note that any 3-Sasakian manifold is Einstein and can be endowed with three 1-forms  $\eta^\alpha$ , three 2-forms  $\omega^\alpha$ , a 3-form  $P$  and a 4-form  $Q$  with  $\alpha = 1, 2, 3$  [1, 2]. In a local orthonormal co-frame  $\{e^{\hat{a}}\}$  where  $\{\hat{a}\} = \{a, 4n+\alpha\}$ , these forms can be written as

$$\eta^\alpha = -e^{4n+\alpha}, \quad (2.18)$$

$$\omega^1 = \sum_{j=1}^n (e^{4j-3} \wedge e^{4j} + e^{4j-2} \wedge e^{4j-1}), \quad (2.19)$$

$$\omega^2 = \sum_{j=1}^n (-e^{4j-3} \wedge e^{4j-1} + e^{4j-2} \wedge e^{4j}), \quad (2.20)$$

$$\omega^3 = \sum_{j=1}^n (e^{4j-3} \wedge e^{4j-2} + e^{4j-1} \wedge e^{4j}), \quad (2.21)$$

$$P = \frac{1}{3} (\sum_\alpha \eta^\alpha \wedge \omega^\alpha + \eta^{123}) = -\frac{1}{3} (\sum_\alpha \omega^\alpha \wedge e^{4n+\alpha} + e^{4n+1} \wedge e^{4n+2} \wedge e^{4n+3}), \quad (2.22)$$

$$Q = \frac{1}{6} \sum_\alpha \omega^\alpha \wedge \omega^\alpha. \quad (2.23)$$

The forms  $e^{4n+\alpha}$  and  $\omega^\alpha$  satisfy the differential identities

$$de^{4n+\alpha} = -\varepsilon_{\beta\gamma}^\alpha e^{4n+\beta} \wedge e^{4n+\gamma} - 2\omega^\alpha, \quad (2.24)$$

$$d\omega^\alpha = -2\varepsilon_{\beta\gamma}^\alpha e^{4n+\beta} \wedge \omega^\gamma, \quad (2.25)$$

where  $\varepsilon$  is the Levi-Civita tensor. The torsion  $T$  of the canonical  $sp(n)$ -valued connection  $\Gamma$  on any 3-Sasakian manifold takes the form (see e.g. [9])

$$T^a = \frac{3}{2} P_{\hat{a}\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}} \quad \text{and} \quad T^\alpha = 3P_{4n+\alpha\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}}. \quad (2.26)$$

$T$  is not totally antisymmetric for the Einstein metric on  $\mathcal{M}$ .

**KT structure on sine-cones.** A well-known theorem [2] states that, if  $(\mathcal{M}, g)$  is a  $k$ -dimensional Einstein manifold with Einstein constant  $k-1$ , then the sine-cone  $(C(\mathcal{M}), \tilde{g})$  over  $\mathcal{M}$  with the metric (2.10) for  $\Lambda = 1$  is Einstein with Einstein constant  $k$ . Here we will show that the sine-cone over any Sasaki-Einstein manifold is not only Einstein but also carries a Kähler-torsion structure.

Consider the cylinder

$$\text{Cyl}(\mathcal{M}) = (\mathbb{R} \times \mathcal{M}, g_{\text{cyl}}) \quad \text{with} \quad g_{\text{cyl}} = \delta_{\hat{a}\hat{b}} e^{\hat{a}} \otimes e^{\hat{b}} + e^{2n+2} \otimes e^{2n+2}, \quad (2.27)$$

where  $\{\hat{a}\} = \{a, 2n+1\}$  with  $a = 1, \dots, 2n$ , and compare it to the sine-cone

$$M^{2n+2} := C_\Lambda(\mathcal{M}) = ((0, \Lambda\pi) \times \mathcal{M}, \tilde{g}) \quad (2.28)$$

parametrized via

$$e^{2n+2} = d\tau = \frac{d\varphi}{\sin \varphi}, \quad \varphi = \frac{r}{\Lambda}, \quad \tau \in \mathbb{R}, \quad \varphi \in (0, \pi). \quad (2.29)$$

Then the local basis  $\{\tilde{e}^{\hat{a}}, \tilde{e}^{2n+2}\}$  on the latter is defined as

$$\tilde{e}^{\hat{a}} = \Lambda \sin \varphi e^{\hat{a}} \quad \text{and} \quad \tilde{e}^{2n+2} = \Lambda \sin \varphi e^{2n+2} = dr, \quad (2.30)$$

and its metric reads

$$\tilde{g} = \delta_{\hat{a}\hat{b}} \tilde{e}^{\hat{a}} \otimes \tilde{e}^{\hat{b}} + \tilde{e}^{2n+2} \otimes \tilde{e}^{2n+2} . \quad (2.31)$$

Let us also introduce the 2-form

$$\tilde{\omega} := \Lambda^2 \sin^2 \varphi (\omega + e^{2n+1} \wedge e^{2n+2}) , \quad (2.32)$$

where  $\omega$  is the 2-form defined in (2.15) and obeying (2.16). It is easy to check that

$$d\tilde{\omega} = \frac{2 \cos \varphi - 1}{\Lambda \sin \varphi} \tilde{\omega} \wedge \tilde{e}^{2n+2} = -\frac{2}{\Lambda} \tan \frac{\varphi}{2} \tilde{\omega} \wedge \tilde{e}^{2n+2} . \quad (2.33)$$

The canonical almost complex structure  $J$  on  $M^{2n+2}$  is fixed by  $\tilde{g}$  and  $\tilde{\omega}$  via (2.4)–(2.6) but with the range of  $j$  extended to  $n+1$ .

Finally one can define the torsion

$$\tilde{T} := Jd\tilde{\omega} = -\frac{2}{\Lambda} \tan \frac{\varphi}{2} \tilde{\omega} \wedge \tilde{e}^{2n+1} \quad (2.34)$$

which is proportional to  $P$  from (2.15). Since  $\tilde{T}$  is of type (2,1)+(1,2) w.r.t.  $J$ , the almost complex structure  $J$  is integrable, and we obtain a KT structure on the sine-cone (2.28) over any Sasaki-Einstein manifold.

**Conical HKT structure.** The sine-cone

$$M^{4n+4} = C_\Lambda(\mathcal{M}) = ((0, \Lambda\pi) \times \mathcal{M}, \tilde{g}) \quad (2.35)$$

over a  $(4n+3)$ -dimensional 3-Sasakian manifold  $(\mathcal{M}, g)$  with a metric (2.10) is an Einstein manifold since  $(\mathcal{M}, g)$  is Einstein. Since  $(\mathcal{M}, e^{4n+3}, \omega^3 + e^{4n+1} \wedge e^{4n+2}, g)$  is Sasaki-Einstein, the previous subsection applies, and one can introduce a KT structure on  $M^{4n+4}$  by choosing the 2-form

$$\tilde{\omega} := \Lambda^2 \sin^2 \varphi (\omega^3 + e^{4n+1} \wedge e^{4n+2} + e^{4n+3} \wedge e^{4n+4}) \quad (2.36)$$

as in (2.32)–(2.34), where  $\omega^3$  is defined by (2.21) and

$$e^{4n+4} = d\tau , \quad \tau = \log(2 \frac{\Lambda}{r_0} \tan \frac{\varphi}{2}) , \quad \varphi = \frac{r}{\Lambda} . \quad (2.37)$$

Here  $\{e^{\hat{a}}, e^{4n+4}\} = \{e^a, e^{4n+\alpha}, e^{4n+4}\}$  with  $a = 1, \dots, 4n$  is a local basis of one-forms on the cylinder  $\mathcal{M} \times \mathbb{R}$  with the metric

$$g_{\text{cyl}} = \delta_{ab} e^a \otimes e^b + \delta_{\mu\nu} e^{4n+\mu} \otimes e^{4n+\nu} , \quad (2.38)$$

where we introduced the index set  $\{\mu\} = \{\alpha, 4\}$ . Recall that  $e^{\hat{a}}$  and  $\omega^\alpha$ , defined in (2.18)–(2.21), satisfy the identities (2.24) and (2.25).

We have

$$\omega^\alpha = \frac{1}{2} \omega_{ab}^\alpha e^a \wedge e^b \quad \text{for } a = 1, \dots, 4n , \quad (2.39)$$

where the components  $\omega_{ab}^\alpha$  of the 2-forms  $\omega^\alpha$  can be read off from (2.19)–(2.21). For later use we define three more 2-forms,

$$\omega_\perp^\alpha := \frac{1}{2} \eta_{\mu\nu}^\alpha e^{4n+\mu} \wedge e^{4n+\nu} , \quad (2.40)$$

where  $\eta_{\mu\nu}^\alpha$  are the components of the 't Hooft tensor,

$$\eta_{\beta\gamma}^\alpha = \varepsilon_{\beta\gamma}^\alpha \quad \text{and} \quad \eta_{\beta 4}^\alpha = -\eta_{4\beta}^\alpha = \delta_\beta^\alpha . \quad (2.41)$$

Using (2.39)–(2.41), we may introduce on  $M^{4n+4}$  three almost complex structures  $J^\alpha$  with components

$$J_b^{\alpha a} = \omega_{bc}^\alpha \delta^{ca} \quad \text{and} \quad J_{4n+\nu}^{\alpha 4n+\mu} = \eta_{\nu\sigma}^\alpha \delta^{\sigma\mu} . \quad (2.42)$$

It is not difficult to show that

$$J^\alpha J^\beta = -\delta^{\alpha\beta} \text{id} + \varepsilon^{\alpha\beta\gamma} J^\gamma , \quad (2.43)$$

i.e. the three almost complex structures define a quaternionic structure on  $M^{4n+4}$ .

The 2-form  $\tilde{\omega}$  in (2.36) is of type (1,1) w.r.t.  $J^3$ . Differentiating, we find

$$\tilde{T} = J^3 d\tilde{\omega} = -\frac{2}{\Lambda} \tan \frac{\varphi}{2} \tilde{\omega} \wedge e^{4n+3} , \quad (2.44)$$

where we use local co-frame fields

$$\tilde{e}^a = \Lambda \sin \varphi e^a \quad \text{and} \quad \tilde{e}^{4n+\mu} = \Lambda \sin \varphi e^{4n+\mu} \quad (2.45)$$

on  $M^{4n+4} = C_\Lambda(\mathcal{M})$ . Thus,  $M^{4n+4}$  allows for a KT structure.

In order to extend this to a HKT structure on  $M^{4n+4}$ , we need three 2-forms

$$\tilde{\omega}^\alpha = \Lambda^2 \sin^2 \varphi (f_1 \omega^\alpha + f_2 \omega_\perp^\alpha) , \quad (2.46)$$

where  $f_1 = f_1(\varphi)$  and  $f_2 = f_2(\varphi)$  are yet undefined real functions, and  $\omega^\alpha$  and  $\omega_\perp^\alpha$  are given by (2.39) and (2.40), respectively. Taking the exterior derivative of (2.46), we obtain

$$d\tilde{\omega}^\alpha = \Lambda^2 \sin^2 \varphi \left\{ (B_1 \omega^\alpha + \frac{1}{2} B_2 \varepsilon_{\beta\gamma}^\alpha e^{4n+\beta} \wedge e^{4n+\gamma}) \wedge e^{4n+4} + B_3 \varepsilon_{\beta\gamma}^\alpha e^{4n+\beta} \wedge \omega^\gamma \right\} , \quad (2.47)$$

where

$$B_1 = \dot{f}_1 \sin \varphi + 2f_1 \cos \varphi - 2f_2 , \quad (2.48)$$

$$B_2 = \dot{f}_2 \sin \varphi + 2f_2 \cos \varphi - 2f_2 , \quad (2.49)$$

$$B_3 = -2(f_1 - f_2) , \quad (2.50)$$

and the overdot indicates a derivative w.r.t.  $\varphi$ . The definitions of  $J^\alpha$  imply that

$$J^\alpha \omega^\beta = (-1)^{1+\delta_{\alpha\beta}} \omega^\beta \quad \text{and} \quad (2.51)$$

$$\begin{aligned} J^1 e^{4n+1} &= -e^{4n+4} , & J^2 e^{4n+1} &= e^{4n+3} , & J^3 e^{4n+1} &= -e^{4n+2} , \\ J^1 e^{4n+2} &= -e^{4n+3} , & J^2 e^{4n+2} &= -e^{4n+4} , & J^3 e^{4n+2} &= e^{4n+1} , \\ J^1 e^{4n+3} &= e^{4n+2} , & J^2 e^{4n+3} &= -e^{4n+1} , & J^3 e^{4n+3} &= -e^{4n+4} , \\ J^\alpha e^{4n+4} &= e^{4n+\alpha} . \end{aligned} \quad (2.52)$$

In [13] it has been proven that the HKT condition is equivalent to

$$J^1 d\tilde{\omega}^1 = J^2 d\tilde{\omega}^2 = J^3 d\tilde{\omega}^3 = \tilde{T} , \quad (2.53)$$

where  $\tilde{T}$  is the torsion of the  $sp(n+1)$ -valued hyper-Hermitian connection on  $M^{4n+4}$ . Using (2.51) and (2.52) as well as demanding that  $\tilde{T}$  is proportional to  $P$  from (2.22), we obtain from (2.53) the constraints

$$B_1 = B_2 = B_3 \quad (2.54)$$

which are equivalent to the differential equations

$$\dot{f} \sin \varphi + 2f \cos \varphi = 0 \quad \text{and} \quad \dot{f}_2 \sin \varphi + 2f_2 (\cos \varphi - 1) + 2f = 0 \quad \text{for} \quad f := f_1 - f_2 . \quad (2.55)$$

Solutions of these equations can be chosen in the form

$$f = \frac{c}{\sin^2 \varphi} \quad \text{and} \quad f_2 = \frac{2c_2}{\cos^4 \frac{\varphi}{2}} + \frac{c}{\sin^2 \varphi} \quad \Rightarrow \quad f_1 = \frac{2c_2}{\cos^4 \frac{\varphi}{2}} + \frac{2c}{\sin^2 \varphi}, \quad (2.56)$$

where  $c$  and  $c_2$  are yet arbitrary constants of integration. In the limit  $\Lambda \rightarrow \infty$  we would like our HKT space  $M^{4n+4}$  to coincide with the standard hyper-Kähler metric cone  $C(\mathcal{M})$  with vanishing torsion  $\tilde{T} = 0$ . This is achieved for

$$c = \frac{c_1}{\Lambda^3}, \quad (2.57)$$

where  $c_1$  is constant. Then for  $\Lambda \rightarrow \infty$  we get

$$\tilde{\omega}^\alpha \rightarrow 2c_2 r^2 (\omega^\alpha + \omega_\perp^\alpha) =: \hat{\omega}^\alpha \quad \text{with} \quad d\hat{\omega}^\alpha = 0. \quad (2.58)$$

The metric on the HKT manifold  $M^{4n+4}$  with three Hermitian<sup>5</sup> structures (2.46) takes the form

$$\tilde{g} = f_1 \delta_{ab} \tilde{e}^a \otimes \tilde{e}^b + f_2 \delta_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu = f_2 \left( \frac{f_1}{f_2} \delta_{ab} \tilde{e}^a \otimes \tilde{e}^b + \delta_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu \right). \quad (2.59)$$

It is conformally equivalent, with conformal factor  $f_2$ , to the metric on the sine-cone  $C_\Lambda(\mathcal{M})$  over a 3-Sasakian manifold  $\mathcal{M}$ . The simplest case occurs for the choice  $c_1 = 0$  when  $f_1 = f_2$ .

### 3 Instantons on conical KT and HKT manifolds

**Instanton equations.** Let  $\Sigma$  be a differential form of degree  $m-4$  on an  $m$ -dimensional Riemannian manifold  $M$ , and let  $\mathcal{E}$  be a complex vector bundle over  $M$  endowed with a connection  $\mathcal{A}$ . The  $\Sigma$ -anti-self-duality (or instanton) equations are defined as the first-order equations [21]

$$*\mathcal{F} = -\Sigma \wedge \mathcal{F} \quad (3.1)$$

for the connection  $\mathcal{A}$  with curvature  $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ . Here  $*$  is the Hodge duality operator on  $M$ . Taking the exterior derivative of (3.1) and using the Bianchi identity, we obtain

$$d*\mathcal{F} + \mathcal{A} \wedge *\mathcal{F} - (-1)^m *\mathcal{F} \wedge \mathcal{A} + *\mathcal{H} \wedge \mathcal{F} = 0, \quad (3.2)$$

where the 3-form  $\mathcal{H}$  is defined by

$$*\mathcal{H} := d\Sigma. \quad (3.3)$$

The second-order equations (3.2) differ from the standard Yang-Mills equations by the last term involving a 3-form  $\mathcal{H}$ , which can be identified with a totally antisymmetric torsion on  $M$ . This torsion term naturally appears in string-theory compactifications with fluxes [34]. For  $d\Sigma = 0$ , the torsion term vanishes and the instanton equations (3.1) imply the ordinary Yang-Mills equations. The latter also holds true when the instanton solution  $\mathcal{F}$  satisfies  $d\Sigma \wedge \mathcal{F} = 0$  as well, as e.g. on nearly Kähler 6-manifolds, nearly parallel  $G_2$ -manifolds and Sasakian manifolds [9].

The torsionful Yang-Mills equations (3.2) are the variational equations for the action

$$S = - \int_M \text{Tr}(\mathcal{F} \wedge *\mathcal{F} + \mathcal{F} \wedge \mathcal{F} \wedge \Sigma), \quad (3.4)$$

and the instanton equations (3.1) can be derived from this action using a Bogomolny argument. In the case of a closed form  $\Sigma$ , the second term in (3.4) is topological and the torsion (3.3) disappears from (3.2).

---

<sup>5</sup>Note that the conditions (2.53) imply the integrability of the almost complex structures (2.42).



If a manifold is endowed with a 4-form  $Q$ , a natural choice for the  $(m-4)$ -form  $\Sigma$  in the instanton equations (3.1) will be its Hodge dual,  $\Sigma \propto *Q$ . Therefore, on KT manifolds with  $m = 2n+2$  one should take

$$\tilde{Q}_{\text{KT}} = \frac{1}{2} \tilde{\omega} \wedge \tilde{\omega} \quad \text{in} \quad * \mathcal{F} = - * \tilde{Q}_{\text{KT}} \wedge \mathcal{F} . \quad (3.5)$$

On HKT manifolds there exist three 2-forms  $\tilde{\omega}^\alpha$ , from which one can build the 4-form

$$\tilde{Q}_{\text{HKT}} = \frac{1}{6} \tilde{\omega}^\alpha \wedge \tilde{\omega}^\alpha \quad \text{in} \quad * \mathcal{F} = - * \tilde{Q}_{\text{HKT}} \wedge \mathcal{F} . \quad (3.6)$$

**Reduction to matrix equations.** Recall that the instanton equations on the cone  $C(\mathcal{M})$  over Sasaki-Einstein or 3-Sasakian manifolds  $\mathcal{M}$  are equivalent to the equations on the cylinder  $\text{Cyl}(\mathcal{M})$  [9] with the metric

$$\tilde{g} = d\tau^2 + g , \quad (3.7)$$

where  $g$  is the metric on  $\mathcal{M}$  and  $\tau$  is related with  $r$  by (2.11). Let us denote by  $G$  and  $H$  the structure groups of the canonical connection on  $C(\mathcal{M})$  and  $\mathcal{M}$ , respectively. In the KT case we have  $(G, H) = (\text{SU}(n+1), \text{SU}(n))$ , and in the HKT case  $(G, H) = (\text{Sp}(n+1), \text{Sp}(n))$ . As a vector space, the Lie algebra  $\mathfrak{g} = \text{Lie } G$  decomposes into  $\mathfrak{h} = \text{Lie } H$  and its orthogonal complement  $\mathfrak{m}$ ,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} . \quad (3.8)$$

The vector space  $\mathfrak{m}$  can be identified with the linear span of the orthonormal basis  $\{e^{\hat{a}}\}$  on  $T^*\mathcal{M}$ .

In any given irreducible representation  $\rho$  of  $\mathfrak{g}$ , the generators  $I_i$  of  $\mathfrak{h}$  and  $I_{\hat{a}}$  of  $\mathfrak{m}$  obeying the commutation relations

$$[I_i, I_j] = f_{ij}^k I_k , \quad [I_i, I_{\hat{a}}] = f_{i\hat{a}}^{\hat{b}} I_{\hat{b}} \quad \text{and} \quad [I_{\hat{a}}, I_{\hat{b}}] = f_{\hat{a}\hat{b}}^i I_i + f_{\hat{a}\hat{b}}^{\hat{c}} I_{\hat{c}} \quad (3.9)$$

act on a representation space  $V \cong \mathbb{C}^N$ , i.e.  $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ . Consider a complex vector bundle  $\mathcal{E} \rightarrow \text{Cyl}(\mathcal{M})$  such that the fibres are copies of  $V$ . Since  $H$  is a closed subgroup of  $G$ , it also acts on the fibres of  $\mathcal{E}$ , but the restriction of our  $\mathfrak{g}$ -representation  $\rho$  to the subalgebra  $\mathfrak{h}$  in general decomposes into a direct sum of several irreducible  $\mathfrak{h}$ -representations, with the corresponding invariant subspaces comprising  $V$ .

The canonical connection  $\Gamma$  on  $T\mathcal{M}$  is always an instanton [9, 10]. On the bundle  $\mathcal{E}$ , it induces the  $\rho(\mathfrak{h})$ -valued connection (denoted by the same letter)

$$\Gamma := \Gamma^i I_i . \quad (3.10)$$

Its curvature

$$R = d\Gamma + \Gamma \wedge \Gamma = (d\Gamma^i + \frac{1}{2} f_{jk}^i \Gamma^j \wedge \Gamma^k) I_i \quad (3.11)$$

satisfies the instanton equations (3.1) [9, 10].

Let us consider some matrix-valued functions  $X_{\hat{a}}(\tau) \in \text{End}(V)$  and introduce on  $\mathcal{E}$  a  $\rho(\mathfrak{g})$ -valued connection

$$\mathcal{A} := \Gamma + X_{\hat{a}} e^{\hat{a}} . \quad (3.12)$$

For  $X_{\hat{a}}$  depending on all coordinates of  $\text{Cyl}(\mathcal{M})$ , this is the general form of a connection on the bundle  $\mathcal{E} \rightarrow \text{Cyl}(\mathcal{M})$ . Below, we shall impose independence of  $X_{\hat{a}}$  on the coordinates of  $\mathcal{M}$  and certain equivariance conditions, which will reduce (3.1) to ordinary differential equations for  $X_{\hat{a}}$ .

Recall that

$$de^{\hat{a}} = -\Gamma_{\hat{b}}^{\hat{a}} \wedge e^{\hat{b}} + T^{\hat{a}} = -\Gamma^i f_{i\hat{b}}^{\hat{a}} \wedge e^{\hat{b}} + \frac{1}{2} T_{\hat{b}\hat{c}}^{\hat{a}} e^{\hat{b}} \wedge e^{\hat{c}} , \quad (3.13)$$

where  $f_{i\hat{b}}^{\hat{a}}$  are the structure constants from (3.9). From (3.12) and (3.13) it follows that

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = R + \frac{1}{2} ([X_{\hat{a}}, X_{\hat{b}}] + T_{\hat{a}\hat{b}}^{\hat{c}} X_{\hat{c}}) e^{\hat{a}} \wedge e^{\hat{b}} + \dot{X}_{\hat{a}} d\tau \wedge e^{\hat{a}} + \Gamma^i \wedge e^{\hat{a}} ([I_i, X_{\hat{a}}] - f_{i\hat{a}}^{\hat{b}} X_{\hat{b}}) , \quad (3.14)$$

where  $R$  is given in (3.11) and  $\dot{X}_{\hat{a}} \equiv \frac{d}{d\tau} X_{\hat{a}}$ . In [27] it was shown that  $\mathcal{F}$  solves the instanton equations (3.1) if the following matrix equations hold:

$$[I_i, X_{\hat{a}}] = f_{i\hat{a}}^{\hat{b}} X_{\hat{b}}, \quad (3.15)$$

$$[X_{\hat{a}}, X_{\hat{b}}] + T_{\hat{a}\hat{b}}^{\hat{c}} X_{\hat{c}} = N_{\hat{a}\hat{b}}^{\hat{c}} \dot{X}_{\hat{c}} + f_{\hat{a}\hat{b}}^i N_i(\tau). \quad (3.16)$$

Here  $N_{\hat{a}\hat{b}}^{\hat{c}}$  is some constant tensor which we shall specify below for each case, and  $N_i$  are some  $\rho(\mathfrak{h})$ -valued functions defined by (3.16) after resolving the algebraic constraints (3.15) and substituting their solutions  $X_{\hat{a}}$  into (3.16). For  $X_{\hat{a}}$  satisfying (3.15) and (3.16), we have

$$\mathcal{F} = R + \frac{1}{2} N_i f_{\hat{a}\hat{b}}^i e^{\hat{a}} \wedge e^{\hat{b}} + \dot{X}_{\hat{a}} (d\tau \wedge e^{\hat{a}} + \frac{1}{2} N_{\hat{b}\hat{c}}^{\hat{a}} e^{\hat{b}} \wedge e^{\hat{c}}), \quad (3.17)$$

where the term with  $f_{\hat{a}\hat{b}}^i$  satisfies (3.1) all by itself (as does  $R$ ) due to the properties of the coset  $G/H$ , and the term proportional to  $\dot{X}_{\hat{a}}$  solves (3.1) after the proper choice of  $N_{\hat{b}\hat{c}}^{\hat{a}}$  to be specified below.

**Instantons on conical KT manifolds.** Consider the sine-cone  $M^{2n+2} = C_{\Lambda}(\mathcal{M})$  over a Sasaki-Einstein manifold  $\mathcal{M}$  of dimension  $2n+1$ . The geometry of  $\mathcal{M}$  and  $M^{2n+2}$  has been discussed in Section 2. In this case, we have  $\{\hat{a}\} = \{a, 2n+1\}$  with  $a = 1, \dots, 2n$ ,  $G = \text{SU}(n+1)$ ,  $H = \text{SU}(n)$  and

$$su(n+1) = su(n) \oplus \mathfrak{m}. \quad (3.18)$$

The sine-cone  $M^{2n+2}$  is conformally equivalent to the cylinder  $\text{Cyl}(\mathcal{M})$  with the local basis 1-forms  $e^{\hat{a}}$  and  $e^{2n+2}$ . The group  $\text{SO}(2n+2)$  acts on the tangent spaces of both  $\text{Cyl}(\mathcal{M})$  and  $C_{\Lambda}(\mathcal{M})$ . We have

$$so(2n+2) = su(n+1) \oplus u(1) \oplus \mathcal{P}, \quad (3.19)$$

so the space of antisymmetric  $(2n+2) \times (2n+2)$  matrices can be split into three mutually orthogonal subspaces, which defines  $\mathcal{P}$ . The  $su(n)$  subspace contains the first two terms in (3.17), which are thus  $\Sigma$ -anti-self-dual. Using the explicit form of the projector from  $so(2n+2)$  to  $su(n+1)$  [17], one can show that the subspace  $\mathfrak{m}$  in (3.18) is spanned by the 2-forms (regarded as antisymmetric matrices)

$$e^{2n+1} \wedge e^{2n+2} - \frac{1}{2n} \omega_{ab} e^a \wedge e^b \quad \text{and} \quad e^a \wedge e^{2n+2} - J_b^a e^b \wedge e^{2n+1}, \quad (3.20)$$

which satisfy the instanton equations (3.1). Here  $\omega_{ab}$  and  $J_b^a$  are as defined in Section 2. Hence, if we choose  $N_{\hat{b}\hat{c}}^{\hat{a}}$  in such a way that the last term in (3.17) becomes a linear combination of the 2-forms (3.20), then  $\mathcal{F}$  from (3.17) will also solve the instanton equations (3.1).

From (2.17), (2.29), (3.17) and (3.20) we finally obtain

$$T_{b2n+1}^a = -\frac{n+1}{n} J_b^a = -f_{b2n+1}^a \quad \text{and} \quad T_{ab}^{2n+1} = 2P_{ab2n+1} = -2\omega_{ab} = -f_{ab}^{2n+1}, \quad (3.21)$$

$$N_{b2n+1}^a = J_b^a = \frac{n}{n+1} f_{b2n+1}^a \quad \text{and} \quad N_{ab}^{2n+1} = \frac{1}{n} \omega_{ab} = \frac{1}{2n} f_{ab}^{2n+1}. \quad (3.22)$$

Substituting (3.21) and (3.22) into (3.15) and (3.16), we arrive at

$$[I_i, X_a] = f_{ia}^b X_b \quad \text{and} \quad [I_i, X_{2n+1}] = 0, \quad (3.23)$$

$$[X_a, X_b] = f_{ab}^{2n+1} (X_{2n+1} + \frac{1}{2n} \dot{X}_{2n+1}) + f_{ab}^j N_j(\tau), \quad (3.24)$$

$$[X_{2n+1}, X_a] = f_{2n+1a}^b (X_b + \frac{n}{n+1} \dot{X}_b). \quad (3.25)$$

The task now is to find solutions to the above matrix equations (3.23)–(3.25). The simplest choice is

$$X_a(\tau) = \psi(\tau) I_a \quad \text{and} \quad X_{2n+1}(\tau) = \chi(\tau) I_{2n+1} \quad (3.26)$$

introducing two functions  $\psi$  and  $\chi$  of  $\tau$ , which is related with  $r$  via (2.11). For this choice, the conditions (3.23) are fulfilled, and we get  $N_i = \psi^2 I_i$ . This reduces (3.24)–(3.25) to the equations

$$\dot{\psi} = \frac{n+1}{n} \psi (\chi - 1) = -\frac{\partial W}{\partial \psi} \quad \text{and} \quad \dot{\chi} = 2n(\psi^2 - \chi) = -\lambda^2 \frac{\partial W}{\partial \chi}, \quad (3.27)$$

which agrees with (4.21) and (4.22) of [9] for the metric cone. Here,  $\lambda = 2n/\sqrt{n+1}$ , and we introduced the flow potential

$$W(\psi, \chi) = \frac{n+1}{2n} (\psi^2 + \frac{1}{2}\chi^2 - \psi^2 \chi) \quad (3.28)$$

for the variables  $\psi$  and  $\tilde{\chi} = \chi/\lambda$ , so that the second equation in (3.27) reads  $\dot{\tilde{\chi}} = -\partial W/\partial \tilde{\chi}$ . For an instanton solution, we need  $\psi$  and  $\chi$  to remain bounded for all  $\tau \in \mathbb{R}$ . This requires the flow to start and end in a critical point of  $W$ . Modulo the obvious reflection symmetry  $\psi \rightarrow -\psi$ , the critical points of  $W$  are

$$\text{the local minimum } (\psi, \chi) = (0, 0) \quad \text{and} \quad \text{the saddle point } (\psi, \chi) = (1, 1), \quad (3.29)$$

and the flow trajectory connecting them is a separatrix for the vector field  $\nabla W$ . It is given by

$$2\psi(1-\chi) d\chi = \lambda^2 (\chi - \psi^2) d\psi, \quad (3.30)$$

which admits analytic solutions only for

$$n = 1 : \chi = \psi \quad \text{and} \quad n \rightarrow \infty : \chi = \psi^2. \quad (3.31)$$

These and the numerical solutions for  $n = 2, 4, 8$  have been plotted in Figure 1 of [9]. Here, we display the equipotential lines of the flow potential  $W$  and for  $n=2$  the corresponding streamlines. The unique bounded solution to (3.27) with  $\tau$  defined by (2.11) yields a Yang-Mills instanton after substituting  $X_a = \psi I_a$  and  $X_{2n+1} = \chi I_{2n+1}$  into (3.12) and (3.17).

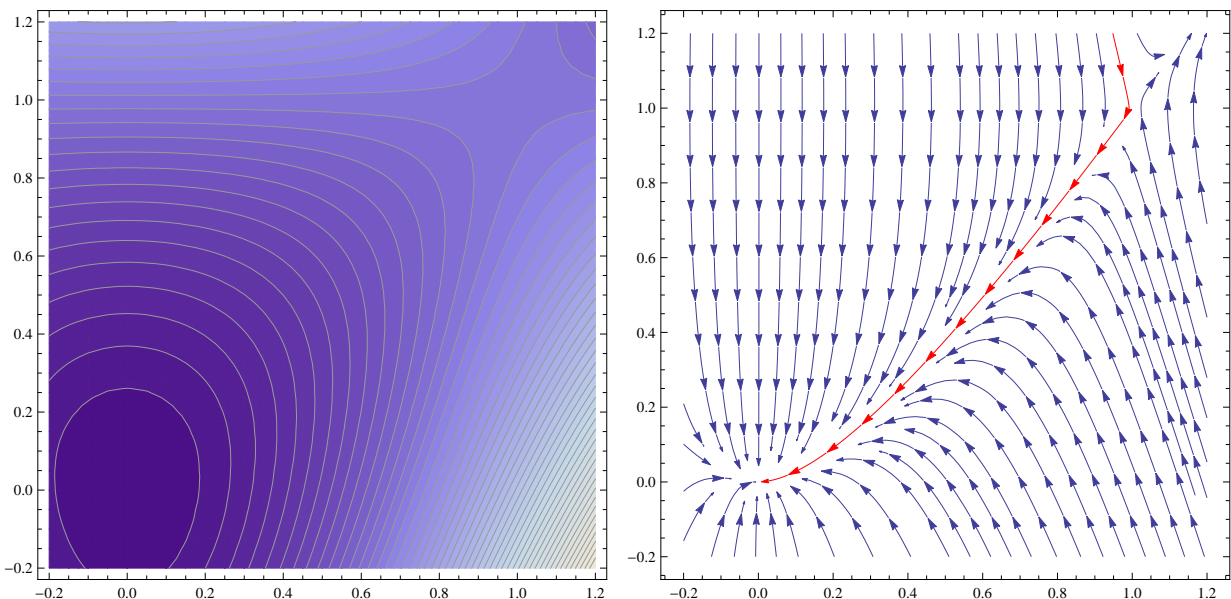


Figure 1: equipotential lines for  $W(\psi, \chi)$  (left) and streamlines for  $n=2$  (right)

**Instantons on conical HKT manifolds.** In Section 2 we have shown that, for  $\mathcal{M}$  being 3-Sasakian of dimension  $4n+3$ , on  $M^{4n+4} = C_\Lambda(\mathcal{M})$  one can introduce a HKT structure with a

metric conformally equivalent to the metric on the cylinder  $\text{Cyl}(\mathcal{M})$ . For this reason it suffices to investigate the instanton equation (3.1) on  $\text{Cyl}(\mathcal{M})$ . In the 3-Sasakian case,  $\{\hat{a}\} = \{a, 4n+\alpha\}$  with  $a = 1, \dots, 4n$  and  $\alpha = 1, 2, 3$ , and we have  $G = \text{Sp}(n+1)$  and  $H = \text{Sp}(n)$ , i.e.

$$sp(n+1) = sp(n) \oplus \mathfrak{m} . \quad (3.32)$$

The group  $\text{SO}(4n+4)$  acts on tangent spaces of both  $\text{Cyl}(\mathcal{M})$  and  $M^{4n+4}$ . One gets

$$so(4n+4) = sp(n+1) \oplus sp(1) \oplus \mathcal{P}' ,$$

and the  $\Sigma$ -anti-self-dual 2-forms reside in the  $sp(n+1)$  subspace of the space of antisymmetric  $(4n+4) \times (4n+4)$  matrices [17]. Now the first two terms in (3.17) sit in  $sp(n) \subset sp(n+1) \subset so(4n+4)$  and, therefore, satisfy the instanton equation (3.1). Employing the explicit form of the projector from  $so(4n+4)$  to  $sp(n+1)$  [17], one can show that the 2-forms

$$\begin{aligned} f_2 \bar{\eta}_{\mu\nu}^\alpha e^{4n+\mu} \wedge e^{4n+\nu} &= f_2 (\varepsilon_{\beta\gamma}^\alpha e^{4n+\beta} \wedge e^{4n+\gamma} - 2e^{4n+\alpha} \wedge e^{4n+4}) \\ \text{and} \quad (f_1 f_2)^{1/2} (e^a \wedge e^{4n+4} + J_b^{\alpha a} e^b \wedge e^{4n+\alpha}) \end{aligned} \quad (3.33)$$

are  $\Sigma$ -anti-self-dual and form a basis of the subspace  $\mathfrak{m}$  in (3.32). Here  $e^{4n+4} = d\tau$ , and  $J_b^{\alpha a}$  as well as the functions  $f_1, f_2$  have been defined in Section 2. From (2.26), (3.17) and (3.33) it follows that the non-vanishing components are given by

$$T_{ab}^{4n+\alpha} = -f_{ab}^{4n+\alpha} , \quad T_{a4n+\beta}^b = -f_{a4n+\beta}^b \quad \text{and} \quad T_{4n+\beta4n+\gamma}^{4n+\alpha} = -f_{4n+\beta4n+\gamma}^{4n+\alpha} , \quad (3.34)$$

$$N_{b4n+\alpha}^a = -J_b^{\alpha a} = -\omega_{ab}^\alpha = f_{b4n+\alpha}^a , \quad N_{4n+\beta4n+\gamma}^{4n+\alpha} = \varepsilon_{\beta\gamma}^\alpha = \frac{1}{2} f_{4n+\beta4n+\gamma}^{4n+\alpha} . \quad (3.35)$$

Substituting (3.34) and (3.35) into (3.15) and (3.16), we arrive at

$$[I_i, X_a] = f_{ia}^b X_b \quad \text{and} \quad [I_i, X_{4n+\alpha}] = 0 , \quad (3.36)$$

$$[X_a, X_b] = f_{ab}^{4n+\alpha} X_{4n+\alpha} + f_{ab}^i N_i , \quad (3.37)$$

$$[X_a, X_{4n+\beta}] = f_{a4n+\beta}^b (X_b + \dot{X}_b) , \quad (3.38)$$

$$[X_{4n+\alpha}, X_{4n+\beta}] = f_{4n+\alpha4n+\beta}^{4n+\gamma} (X_{4n+\gamma} + \frac{1}{2} \dot{X}_{4n+\gamma}) , \quad (3.39)$$

again independent of the functions  $f_1$  and  $f_2$ . If we choose the simplest ansatz

$$X_a(\tau) = \psi(\tau) I_a \quad \text{and} \quad X_{4n+\alpha}(\tau) = \chi(\tau) I_{4n+\alpha} , \quad (3.40)$$

then (3.36) will be satisfied identically. From (3.37) we obtain  $N_i = \psi^2 I_i$ , and (3.38) and (3.39) reduce to

$$\dot{\psi} = \psi(\chi - 1) \quad \text{and} \quad \dot{\chi} = 2\chi(\chi - 1) \quad \text{as well as} \quad \chi = \psi^2 , \quad (3.41)$$

which is the  $n \rightarrow \infty$  limit of (3.27) and coincides with (4.31)-(4.33) of [9] for the metric cone. The equations decouple to

$$\dot{\psi} = \psi(\psi + 1)(\psi - 1) \quad \text{and} \quad \dot{\chi} = 2\chi(\chi - 1) , \quad (3.42)$$

whose only bounded solution is

$$\chi = \psi^2 = \frac{1}{2}(1 - \tanh(\tau - \tau_0)) . \quad (3.43)$$

Substituting  $X_a = \psi I_a$  and  $X_{4n+\alpha} = \chi I_{4n+\alpha}$  into (3.12) and (3.17), one obtains Yang-Mills instanton configurations on the HKT manifold  $M^{4n+4}$  after using (2.11) and the relations (2.45) between the co-frame fields on  $M^{4n+4}$  and the cylinder. More general instanton solutions may be obtained by considering more general ansätze for the matrices  $X_{\hat{a}}$ .

## 4 Conclusions

A Killing spinor on a Riemannian manifold  $\mathcal{M}$  is a spinor field  $\epsilon$  obeying the equation  $\nabla_{\hat{a}}\epsilon = i\lambda\gamma_{\hat{a}}\epsilon$ , where  $\nabla_{\hat{a}}$  is the spinor covariant derivative,  $\gamma_{\hat{a}}$  are Clifford  $\gamma$ -matrices and  $\lambda$  is a constant. Manifolds with real Killing spinors often occur in string-theory compactifications. All these manifolds feature connections with non-vanishing torsion and admit a non-integrable  $H$ -structure, i.e. a reduction of the structure group  $SO(m)$  of the tangent bundle  $T\mathcal{M}$  to  $H \subset SO(m)$ . The metric cone  $C(\mathcal{M})$  over any such manifold  $\mathcal{M}$  has a special (reduced) holonomy group  $G \subset SO(m+1)$  and a Killing spinor  $\epsilon$  with  $\lambda = 0$  (called parallel spinor). These manifolds were classified in [35], and, besides the round spheres, they are the

- nearly Kähler 6-manifolds, with  $H = SU(3)$  and  $G = G_2$
- nearly parallel 7-manifolds, with  $H = G_2$  and  $G = Spin(7)$
- Sasaki-Einstein  $(2n+1)$ -manifolds, with  $H = SU(n)$  and  $G = SU(n+1)$
- 3-Sasakian  $(4n+3)$ -manifolds, with  $H = Sp(n)$  and  $G = Sp(n+1)$

Instantons on metric cones  $C(\mathcal{M})$  over the above manifolds  $\mathcal{M}$  were described in [9, 23, 24, 27]. Instantons on sine-cones over nearly Kähler 6-manifolds and nearly parallel 7-manifolds with  $G_2$ -structure were investigated in [26]. Here, we completed this study by describing Yang-Mills instantons on sine-cones over Sasaki-Einstein and 3-Sasakian manifolds. In [9, 10] instantons on metric cones were extended to brane-type solutions of heterotic supergravity. It would be of interest to perform a similar lift of instantons on sine-cones.

## Acknowledgements

This work was supported in part by the Deutsche Forschungsgemeinschaft under the grant LE 838/13 and by the Heisenberg-Landau program.

## References

- [1] C.P. Boyer and K. Galicki, “3-Sasakian manifolds,” *Surveys Diff. Geom.* **7** (1999) 123 [hep-th/9810250].
- [2] C.P. Boyer and K. Galicki, “Sasakian geometry, holonomy, and supersymmetry,” in: *Handbook of Pseudo-Riemannian Geometry and Supersymmetry*, European Mathematical Society Publishing House 2010, pp.39-84 [math/0703231 [math.DG]].
- [3] C.P. Boyer and K. Galicki, *Sasakian geometry*, Oxford University Press, Oxford, 2008.
- [4] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Sasaki-Einstein metrics on  $S^2 \times S^3$ ,” *Adv. Theor. Math. Phys.* **8** (2004) 711 [hep-th/0403002];  
J.P. Gauntlett, D. Martelli, J.F. Sparks and D. Waldram, “A new infinite class of Sasaki-Einstein manifolds,” *Adv. Theor. Math. Phys.* **8** (2006) 987 [hep-th/0403038];  
D. Martelli and J. Sparks, “Toric Sasaki-Einstein metrics on  $S^2 \times S^3$ ,” *Phys. Lett. B* **621** (2005) 208 [hep-th/0505027].
- [5] M. Cvetič, H. Lü, D.N. Page and C.N. Pope, “New Einstein-Sasaki spaces in five and higher dimensions,” *Phys. Rev. Lett.* **95** (2005) 071101 [hep-th/0504225];  
M. Cvetič, H. Lü, D.N. Page and C.N. Pope, “New Einstein-Sasaki and Einstein spaces from Kerr-de Sitter,” *JHEP* **07** (2009) 082 [hep-th/0505223].
- [6] J. Sparks, “Sasaki-Einstein Manifolds,” *Surveys Diff. Geom.* **16** (2011) 265 [arXiv:1004.2461 [math.DG]].
- [7] B.S. Acharya, J.M. Figueroa-O’Farrill, C.M. Hull and B.J. Spence, “Branes at conical singularities and holography,” *Adv. Theor. Math. Phys.* **2** (1999) 1249 [hep-th/9808014].
- [8] P. Koerber, D. Lüst and D. Tsimpis, “Type IIA  $AdS_4$  compactifications on cosets, interpolations and domain walls,” *JHEP* **07** (2008) 017 [arXiv:0804.0614 [hep-th]].
- [9] D. Harland and C. Nölle, “Instantons and Killing spinors,” *JHEP* **03** (2012) 082 [arXiv:1109.3552 [hep-th]].
- [10] K.-P. Gemmer, A.S. Haupt, O. Lechtenfeld, C. Nölle and A.D. Popov, “Heterotic string plus five-brane systems with asymptotic  $AdS_3$ ,” *Adv. Theor. Math. Phys.* **17** (2013) 771 [arXiv:1202.5046 [hep-th]];  
C. Nölle, “Instantons, five-branes and fractional strings,” arXiv:1207.7268 [hep-th].
- [11] P.S. Howe and G. Papadopoulos, “Twistor spaces for HKT manifolds,” *Phys. Lett. B* **379** (1996) 80 [hep-th/9602108].
- [12] G.W. Gibbons, G. Papadopoulos and K.S. Stelle, “HKT and OKT geometries on soliton black hole moduli spaces,” *Nucl. Phys. B* **508** (1997) 623 [hep-th/9706207];  
A. Opfermann and G. Papadopoulos, “Homogeneous HKT and QKT manifolds,” math-ph/9807026;  
J. Gutowski and G. Papadopoulos, “The moduli spaces of world volume brane solitons,” *Phys. Lett. B* **432** (1998) 97 [hep-th/9802186].
- [13] G. Grantcharov and Y.S. Poon, “Geometry of hyper-Kähler connections with torsion,” *Commun. Math. Phys.* **213** (2000) 19 [math/9908015 [math.DG]].
- [14] G. Grantcharov, G. Papadopoulos and Y.S. Poon, “Reduction of HKT structures,” *J. Math. Phys.* **43** (2002) 3766 [math/0201159 [math.DG]].

- [15] M.L. Barberis, “A survey on hyper-Kähler with torsion geometry,”  
 Revista de la Union Math. Argentina, **49** (2008) 121;  
 M. Fernandes, A. Fino, L. Ugarte and R. Villacampa,  
 “HKT structures from almost contact manifolds,”  
 XIX Int. Fall Workshop on Geometry and Physics, AIP Conf. Proc. **1360** (2011) 27-38.
- [16] E. Corrigan, C. Devchand, D.B. Fairlie and J. Nuyts, “First order equations for gauge fields  
 in spaces of dimension greater than four,” Nucl. Phys. B **214** (1983) 452;  
 R.S. Ward, “Completely solvable gauge field equations in dimension greater than four,”  
 Nucl. Phys. B **236** (1984) 381.
- [17] T.A. Ivanova and A.D. Popov, “(Anti)self-dual gauge fields in dimension  $d \geq 4$ ,”  
 Theor. Math. Phys. **94** (1993) 225.
- [18] L. Baulieu, H. Kanno and I.M. Singer, “Special quantum field theories in eight and other  
 dimensions,” Commun. Math. Phys. **194** (1998) 149 [hep-th/9704167];  
 M. Blau and G. Thompson, “Euclidean SYM theories by time reduction and special holonomy  
 manifolds,” Phys. Lett. B **415** (1997) 242 [hep-th/9706225];  
 B.S. Acharya, J.M. Figueroa-O’Farrill, B.J. Spence and M. O’Loughlin, “Euclidean D-branes  
 and higher-dimensional gauge theory,” Nucl. Phys. B **514** (1998) 583 [hep-th/9707118].
- [19] S.K. Donaldson, “Infinite determinants, stable bundles and curvature,”  
 Duke Math. J. **54** (1987) 231;  
 K.K. Uhlenbeck and S.-T. Yau, “On the existence of Hermitian-Yang-Mills connections on  
 stable bundles over compact Kähler manifolds,” Commun. Pure Appl. Math. **39** (1986) 257.
- [20] S.K. Donaldson and R.P. Thomas, “Gauge theory in higher dimensions,”  
 in: The Geometric Universe, Oxford University Press, Oxford, 1998;  
 S.K. Donaldson and E. Segal, “Gauge theory in higher dimensions II”,  
 Surveys Diff. Geom. **16** (2011) 1 [arXiv:0902.3239 [math.DG]].
- [21] G. Tian, “Gauge theory and calibrated geometry,”  
 Ann. Math. **151** (2000) 193 [math/0010015 [math.DG]].
- [22] D.B. Fairlie and J. Nuyts, “Spherically symmetric solutions of gauge theories in eight dimen-  
 sions,” J. Phys. A **17** (1984) 2867;  
 S. Fubini and H. Nicolai, “The octonionic instanton,” Phys. Lett. B **155** (1985) 369;  
 T.A. Ivanova and A.D. Popov, “Self-dual Yang-Mills fields in  $d=7,8$ , octonions and Ward  
 equations,” Lett. Math. Phys. **24** (1992) 85.
- [23] T.A. Ivanova, O. Lechtenfeld, A.D. Popov and T. Rahn, “Instantons and Yang-Mills flows on  
 coset spaces,” Lett. Math. Phys. **89** (2009) 231 [arXiv:0904.0654 [hep-th]];  
 D. Harland, T.A. Ivanova, O. Lechtenfeld and A.D. Popov,  
 “Yang-Mills flows on nearly Kähler manifolds and  $G_2$ -instantons,”  
 Commun. Math. Phys. **300** (2010) 185 [arXiv:0909.2730 [hep-th]].
- [24] I. Bauer, T.A. Ivanova, O. Lechtenfeld and F. Lubbe, “Yang-Mills instantons and dyons on  
 homogeneous  $G_2$ -manifolds,” JHEP **10** (2010) 044 [arXiv:1006.2388 [hep-th]];  
 A.S. Haupt, T.A. Ivanova, O. Lechtenfeld and A.D. Popov,  
 “Chern-Simons flows on Aloff-Wallach spaces and Spin(7)-instantons,”  
 Phys. Rev. D **83** (2011) 105028, [arXiv:1104.5231 [hep-th]].
- [25] F.P. Correia, “Hermitian Yang-Mills instantons on Calabi-Yau cones,”  
 JHEP **12** (2009) 004 [arXiv:0910.1096 [hep-th]];

- F.P. Correia, “Hermitian Yang-Mills instantons on resolutions of Calabi-Yau cones,”  
*JHEP* **02** (2011) 054 [arXiv:1009.0526 [hep-th]].
- [26] K.-P. Gemmer, O. Lechtenfeld, C. Nölle and A.D. Popov, “Yang-Mills instantons on cones and sine-cones over nearly Kähler manifolds,” *JHEP* **09** (2011) 103 [arXiv:1108.3951 [hep-th]].
- [27] T.A. Ivanova and A.D. Popov, “Instantons on special holonomy manifolds,”  
*Phys. Rev. D* **85** (2012) 105012 [arXiv:1203.2657 [hep-th]].
- [28] T. Walpuski, “ $G_2$ -instantons on generalised Kummer constructions,”  
*Geom. Topol.* **17** (2013) 2345 [arXiv:1109.6609 [math.DG]];  
A. Clarke, “Instantons on the exceptional holonomy manifolds of Bryant and Salamon,”  
*J. Geom. Phys.* **82** (2014) 84 [arXiv:1308.6358 [math.DG]].
- [29] J.A. Wolf, *Spaces of constant scalar curvature*, McGraw-Hill, New York, 1967;  
J.A. Wolf and A. Gray, “Homogeneous spaces defined by Lie group automorphisms I,II,”  
*J. Diff. Geom.* **2** (1968) 77, 115.
- [30] A. Gray, “Nearly Kähler manifolds,” *J. Diff. Geom.* **4** (1970) 283.
- [31] P. Gauduchon, “Hermitian connections and Dirac operators,”  
*Bollettino U.M.I.* **11B** (1977) 257;  
J.M. Bismut, “A local index theorem for non-Kähler manifolds,” *Math. Ann.* **284** (1989) 681.
- [32] A. Fino, M. Parton and S. Salamon, “Families of strong KT structures in six dimensions,”  
*Comment. Math. Helv.* **79** (2004) 317;  
M. Fernandez, A. Fino, L. Ugarte and R. Villacampa, “Strong Kähler with torsion structures  
from almost contact manifolds,” *Pacific J. Math.* **249** (2011) 49.
- [33] J. Gutowski, S. Ivanov and G. Papadopoulos, “Deformations of generalized calibrations and  
compact non-Kähler manifolds with vanishing first Chern class,”  
*Asian J. Math.* **7** (2003) 39 [math/0205012 [math.DG]];  
D. Grantcharov, G. Grantcharov and Y.S. Poon, “Calabi-Yau connections with torsion on toric  
bundles,” *J. Diff. Geom.* **78** (2008) 13 [math/0306207 [math.DG]].
- [34] M. Graña, “Flux compactifications in string theory: A comprehensive review,”  
*Phys. Rept.* **423** (2006) 91 [hep-th/0509003];  
M.R. Douglas and S. Kachru, “Flux compactification,”  
*Rev. Mod. Phys.* **79** (2007) 733 [hep-th/0610102];  
R. Blumenhagen, B. Kors, D. Lüst and S. Stieberger, “Four-dimensional string compactifica-  
tions with D-branes, orientifolds and fluxes,” *Phys. Rept.* **445** (2007) 1 [hep-th/0610327].
- [35] C. Bär, “Real Killing spinors and holonomy,” *Commun. Math. Phys.* **154** (1993) 509.