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Albert Lautman: Dialectics in mathematics

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Albert Lautman (1908–1944) is a rare example of a twentieth-century philosopher whose engagement with contemporary mathematics goes beyond the 'foundational' areas of mathematical logic and set theory. He insists that (what were in his day) the new mathematics of topology, abstract algebra, class field theory and analytic number theory have a philosophical significance that distinguishes them from the mathematics of earlier eras. Specifically, these new areas of mathematics reveal underlying dialectical structures not found in earlier mathematics. In a series of short papers and two longer theses (*Essay on the unity of the mathematical sciences in their current development* and *Essay on the notions of structure and existence in mathematics*)[1], Lautman argues this claim from a philosophical perspective rooted in certain of the later dialogues of Plato. However, Lautman was not satisfied with Plato's conception of the relation between dialectical Ideas and the matter in which they are realised. In one of his last papers, *New research on the dialectical structure of mathematics*[2], Lautman bolsters his Platonism with an appeal to Heidegger's 'ontological' distinction between phenomenology and science.[3] We may therefore regard this paper as the most advanced expression available of Lautman's philosophy of mathematics.

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[1] Henceforth, page numbers refer to the 2006 Vrin edition of Lautman's complete works, (Lautman, 2006).

[2] (Lautman, 2006, pp. 235–257); this paper was first published in 1939 in a series edited by Jean Cavaillès and Raymond Aron.

[3] As expressed in Heidegger's 1928 lecture *Vom Wesen des Grundes*. Quotations here are from McNeill's 1998 translation *On the Essence of Ground*. Lautman quotes Corbin's 1938 French translation.

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In this paper, I shall first explore Lautman's conception of dialectics by a consideration of his references to Plato and Heidegger. I shall then compare the dialectical structures that he found in contemporary mathematics with the model that emerges from his philosophical sources. I shall argue that the structures that he discovered in mathematics are richer than his Platonist model suggests, and that Heidegger's 'ontological' distinction is less useful than Lautman seemed to believe.

1 Plato

In his major case studies, Lautman developed a picture of modern mathematics (that is, mathematics in the early twentieth century) as the expression or realisation of fundamental conceptual oppositions (such as continuous/discontinuous, global/local, finite/infinite, symmetric/anti-symmetric).[4] He referred to the opposing terms as *notions*; dialectical *Ideas* envisage possible relations between such pairs of dialectical notions (Lautman, 2006, pp. 242–243). This terminology is a conscious reference to Plato, and he is careful to distinguish his appeal to Plato from 'Platonism' as philosophers of mathematics usually use the term. In philosophy of mathematics, 'Platonism' usually denotes the view that mathematical objects exist independently of the thought and talk of mathematicians. Lautman insisted that this was a misreading of Plato (Lautman, 2006, p. 230); in any case, this kind of 'Platonism' is not Lautman's view.[5]

Lautman never quotes Plato directly, and he mentions just three Platonic texts: *Philebus*, the *Sophist* (twice), and *Timaeus* (twice). Scholars usually count these among the 'later' dialogues of Plato (though the *Sophist* is continuous with the *Theaetetus* and implicitly refers to the *Parmenides*—both middle period dialogues). What matters for our purpose is that Plato's *theory of forms* is largely absent from his later works. The *Ideas* in the later dialogues are not blueprints for material objects. Similarly, Lautman's mathematical Platonism was not a 'copy-theory'. As he points out, we might think of material reality as inchoate matter somehow shaped into material copies of non-material 'forms', but this model cannot apply to the relation between mathematical theories and the dialectical ideas that (in Lautman's term) *dominate* them (Lautman, 2006, p. 238).

[4]This list is drawn from the two long essays. In *New Research on the dialectical structure of mathematics* he offers a slightly different list of dialectical pairs, "wholes and parts, situational and intrinsic properties, basic domains and objects defined on these domains, formal systems and their models, etc.." (Lautman, 2006, p. 243)

[5]"Dans le débat ouvert entre formalistes et intuitionnistes, [...] les mathématiciens ont pris l'habitude de désigner sommairement sous le nom de platonisme toute philosophie pour laquelle l'existence d'un être mathématique est tenue pour assuré [...] c'est là une connaissance superficielle du platonisme [...]" (Lautman, 2006, p. 230).

1.1 The *Sophist*

In a short paper of 1937 called *L'axiomatique et la méthode de division*[6], Lautman refers to *Philebus* and the *Sophist* together:

> The movement from so-called 'elementary' notions to abstract notions does not [...] appear as the subsumption of the particular under the general, but rather as the division or analysis of a 'mixture' which tends to yield simple notions in which this mixture participates. It is, therefore, not the Aristotelian logic of genus and species at work here, but the Platonic method of division, as taught in the *Sophist* and *Philebus*, in which the unity of Being is a unity of composition and a starting-point in the search for principles that are unified in Ideas.[7]

The *Sophist* is a discussion between a young man, Theaetetus, and a stranger from Elea, "a comrade of the circle of Parmenides and Zeno, and a man very much a philosopher" (216A). The initial question is whether the words 'sophist', 'statesman' and 'philosopher' name one, two or three types of thing, and what that thing is or those things are. The nameless stranger asks for an "interlocutor [who] submits to guidance easily" (217D); Socrates proposes young Theaetetus. Thus, Plato allows the unnamed philosopher to develop his position at length without having to fend off a Socratic interrogation (this is a feature of Plato's later works; in the eponymous dialogue, Timaeus has the floor to himself after the preliminary civilities). Thereafter, Socrates vanishes from the text, so we do not have the luxury of inferring Plato's view from Socrates's words.

The Eleatic philosopher proceeds by division, that is, by making one distinction after another. He illustrates this technique with the term 'angler'. He first distinguishes gathering arts from manufacturing arts; then the gathering arts are divided into trading and 'mastering' or getting the better of; getting the better of divides into competition and hunting; hunting divides according to quarry (animal or other); animals swim or walk; swimming animals divide into water-fowl and fish; fishing divides into trapping (with nets, traps, etc.) and striking; striking divides into striking down with a trident and up with a hook. The resulting tree of categories is his account of 'angler'. He then proceeds to apply the same technique to the term

[6]*Axiomatics and the method of division*; (Lautman, 2006, pp. 69–80).

[7]"Le passage des notions dites 'élémentaires' aux notions abstraits ne se présente donc pas comme une subsomption du particulier sous le général mais comme la division ou l'analyse d'un 'mixte' qui tend à dégager les notions simples auxquelles ce mixte participe. Ce n'est donc pas la logique aristotélicienne, celle des genres et des espèces qui intervient ici, mais la méthode platonicienne de division, telle que l'enseignent le Sophiste et le Philèbe pour laquelle l'unité de l'Être est une unité de composition et un point de départ vers la recherche des principes qui s'unissent dans les Idées" (Lautman, 2006, pp. 78–79).

'sophist', and this discussion occupies the remainder of the dialogue. The Eleatic philosopher develops several different accounts of 'sophist' (231D-E), which leads to a methodological discussion, including a debate about the possibility of numbering non-beings (238B). The discussion refers to itself, because Theodorus introduced the Eleatic stranger as a philosopher, presumably in virtue of his logical technique (253C).[8] But if the method of division turns out to be merely a spurious word-game, then perhaps *he* is a sophist. Certainly, his choices of divided categories seem arbitrary. For example, he might have divided fishing according to whether or not bait is used, in which case trident-fishing and net-fishing would have been divided from angling and the use of baited traps. Young Theaetetus submits to the philosopher's guidance rather too easily, and certainly more easily than Socrates would have done.

Whatever Plato's intent in giving an unnamed, generic Eleatic philosopher an easy ride, Lautman takes the method of division as an unproblematic technique, and makes no mention of its proper companion, the 'method of collection'. In the text immediately before the excerpt quoted above, Lautman runs through a list of mixtures, that is, mathematical items that 'participate' in two heterogeneous categories. Namely: arithmetical equality is the only equivalence relation such that the number of equivalence classes equals the cardinality of the base domain; the idea of multiplication refers both to the creation of arithmetical products and to the idea of operators on a domain; unity can be thought of either as the unit element of a ring of numbers or as the identity element in a domain of operators; the length of a segment depends on the size of the segment but at the same time depends on a convention; absolute value in classical algebra includes the notion of ordering but also the notion of the completeness of a field. He goes on to claim that some of these mixtures (arithmetical equality; multiplication; absolute value) are examples of the dialectical relation between the intrinsic and relational properties of mathematical objects (Lautman, 2006, pp. 78–79). He then suggests that, "the distinction thus established at the heart of a single concept between the intrinsic properties of an object [...] and its potential for action [on other objects] seems to resemble the Platonic distinction between the Same and the Other [...]"[9]. For Lautman, then, these mathematical items (equality, multiplication, unity, length and absolute value) all have, in some sense, one foot in each of two camps. We shall see this pat-

[8] But cf. Trevaskis's argument that there is more to the philosopher's technique than the method of division (Trevaskis, 1967).

[9] "La distinction qui s'ètablit ainsi au sein d'une même notion entre les propriétés intrinsèques d'un être ou d'une notion et ses possibilités d'action nous semble s'apparenter à la distinction platonicienne du Même et de l'Autre qui se retrouvent dans l'unité de l'être" (Lautman, 2006, p. 79). (Translation note: this translation is a little free in order to preserve Lautman's special sense of *notion*).

tern again in the fifth chapter of the *Essay on the notions of structure and existence in mathematics*, in which Lautman explores another collection of mathematical 'mixtures'. Notice, though, that the pairs of notions in this list are not pairs of conceptual opposites. He has this in common with the Eleatic philosopher; swimming is not the opposite of walking, nor is fish the opposite of fowl. The fact that these pairs are *not* conceptual opposites raises the question why the Eleatic philosopher divides categories into pairs (rather than triples, quadruples, *etc.*), with all the resulting awkwardness and arbitrariness. In another late work that Lautman mentions, *Timaeus*, Plato divides living creatures into four classes according to habitat: gods in heaven, birds in the air, land animals and water animals (39–40). Similarly in *Philebus*, when Socrates describes the method of division he requires only that a category be divided into a finite number of sub-categories (16D). The view that dialectics relates notions in pairs is indeed present in the *Sophist*, but only in the figure of the generic Eleatic philosopher. It does not seem to have been Plato's doctrine.

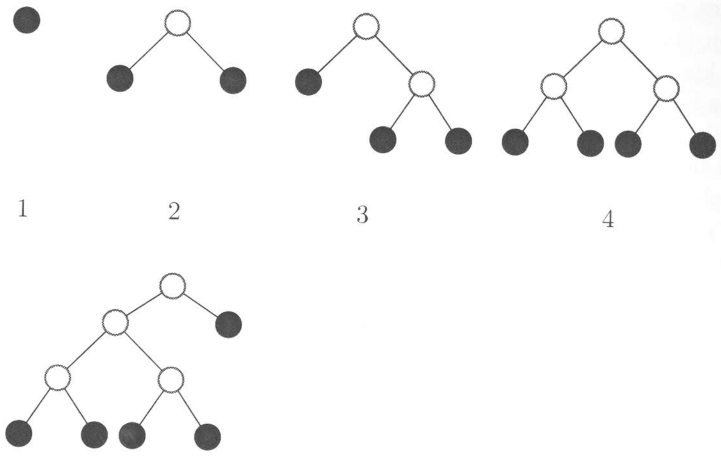

FIGURE 1. Lautman's Tree-diagrams.

The final reference to the *Sophist* is rather indirect. Lautman ends the *Essay on the notions of structure and existence in mathematics* with a gesture towards the thought that there is a developmental order among dialectical Ideas and from Ideas to mathematics. Lautman appeals to the works of Oscar Becker and Julius Stenzel on number in Plato and Aristotle. Lautman supplies a diagram taken from Stenzel (cf. Figure 1). Iterations of the Ideas 'one' and 'pair' produce 'Idea-numbers' (represented by the tree-diagrams), which in turn engender arithmetical numbers (represented by the black dots). Lautman's discussion is confused and inconclusive. He reproduces this diagram in the main text *and* in the footnotes, and mentions some reservations on Becker's part without discussing them (Lautman, 2006, p. 230, Figure 9). In any case, as Lautman acknowledges, Becker and Stenzel were both reading Plato through Aristotle (this was also Heidegger's procedure).[10] Having made this gesture, Lautman then turns to the relationship between mathematics and physics. The brief, inconclusive discussion with its pointlessly repeated diagram suggests some haste and dissatisfaction on Lautman's part.

1.2 *Philebus*

Lautman mentions *Philebus* once, in the quotation given above, as a source for the method of division. In this dialogue, Philebus, one of Socrates's young companions, holds that the good for man is pleasure. Socrates sets out to contest this, and to argue that intelligence (including knowledge and judgement) is better than pleasure. Before proceeding to his argument, Socrates makes a methodological digression. He describes the method of division (16C–17A), and insists that scientific understanding of a topic requires knowledge of the structure generated by successive distinctions. Unlike the Eleatic philosopher, Socrates allows that a category may divide into more than two subclasses; all he insists is that the number of subclasses should be finite (16D).

Early in the dialogue, Socrates points out that a life of pure intellectual activity is not suitable for men (21E). The good life for men must include some sensuous enjoyment as well as intellectual activity—though this being Plato, the intellectual side has priority. The crucial point is that the good for men is a mixture of heterogeneous elements (sensual and intellectual). This would present a paradox, if the method of division had the Aristotelian purpose of establishing a taxonomy. In the 'Aristotelian logic of genus and species', an object that seems to belong to two different species would be a counterexample to the taxonomy (as, for example, the duck-billed platypus threatens the category 'mammal'). In contrast, a Platonic system of Ideas is somehow prior to and independent of the objects that participate in those Ideas. An object can participate in more than one Idea (for example, a physical object might be both red and round). This mixing of Ideas occurs in other later dialogues. It is one of the principal explanatory motifs in the physics of *Timaeus* (34–35, 59–61), and we have already seen it in the *Sophist*.

For our purposes, the significant outcome of *Philebus* is that every human life must embody a mixture of sensual and intellectual goods. Precisely which goods and how they connect will vary from life to life. Perhaps someone's enjoyment of wine will develop an intellectual aspect as connoisseurship. To take a different example, intellectual work may offer some pleasures and satisfactions (though this is not, in Plato, the reason why it is good). One might imagine a life in which intelligence and pleasure were entirely separate departments, though it is hard to imagine desiring such an existence. Every human life will embody this dialectic in some way, and on philosophical examination will disclose it. Lives that lack one or other element must show that lack as an inadequacy or discontent. Indeed, we would need this dialectic of the sensual and the intellectual in order to understand the distempers and changes within a particular life. The background dialectical structure explains why a life given excessively to either sensuous pleasures or intellectual goods would be unsatisfactory.

Philebus, then, gives us an ethical analogy of Lautman's account of dialectical Ideas in mathematics. In Lautman's terminology, pleasure and intelligence are 'notions' and the possibility of relations between them is an 'Idea'. This dialectical structure does not specify which pleasures and thoughts will actually obtain. As Lautman says of mathematical Ideas, "As they are merely sketches of eventual positions, [Ideas] do not necessarily entail the existence of particular beings capable of sustaining the relations that the Ideas outline".[11] These notions come into relation through the interplay of particular thoughts and pleasures, and there is no predicting the detail of that interplay from the bare dialectical structure. The wine connoisseur's knowledge inflects his pleasure in boozing. Pleasure and intelligence relate quite differently (but no less intimately) in the rare but precious moments of insight in the work of a scientist. Similarly, Lautman maintains there is an indefinite variety of ways in which any dialectical relation between notions might manifest itself in actual mathematics, and it is not the business of philosophers to attempt to predict or circumscribe these relations (Lautman, 2006, p. 229).

1.3 *Timaeus*

Lautman's two references to *Timaeus* (Lautman, 2006, pp. 231 & 267) both remind us that for Plato, the creation of a material world is possible only if there is already a 'geometrically ordered receptacle' called 'place'.[12] Crucially, *different* objects may (at different times) occupy the *same* place. Thus, 'place' depends for its intelligibility on an anterior dialectical pair:

[10] Heidegger (1925, p. 8). Lautman was not the only one to find Stenzel's reading of Plato on arithmetic more suggestive than clear. Cf. Cornford (1924); Shorey (1924).

[11] "Étant seulement dessin de positions éventuelles, elles n'entrainent pas forcément l'existence d'êtres susceptibles de soutenir entre eux les relations qu'elles ébauchent" (Lautman, 2006, p. 243).

[12] "[...] le réceptacle d'une qualification géométrique" (Lautman, 2006, p. 231); translation note: the more literal "receptacle of a geometric qualification" makes little sense); "le lieu" (Lautman, 2006, p. 267). Cf. *Timaeus* §§48–9.

same/other. As we saw above, Lautman regards 'the distinction [...] between the intrinsic properties of an object [...] and its potential for action' on other objects as an expression of the same/other relation. In both cases, reference to Timaeus enables Lautman to shift from philosophy of mathematics to philosophy of physics. Lautman argues that the natural world is mathematically intelligible because the same dialectical structures underlie both physics and mathematics. He offers enantiomorphic crystals as an example of a physical phenomenon in which dialectical opposites (in this case, symmetry and dissymmetry) are 'mixed'. (This paper will not further discuss Lautman's philosophy of physics.) Here, as in his allusion to Stenzel's work on number, Lautman is trying to illustrate his thought that the intelligibility of mathematics and physics requires a prior dialectical order. In both cases, his exposition stumbles over Plato's inability to say what 'dialectical priority' means.

For Lautman, then, the method of division reveals dialectical 'notions' (in his special sense of the word), and with them the Ideas of relations between these notions. However, Lautman does not offer sequences of distinctions. His notions do not form tree-shaped accounts like those of the philosopher in the *Sophist*. As the quotation at the head of this section suggests, what he takes from these later dialogues is the thought that a particular can participate in heterogeneous categories simultaneously. In some of his examples, the notions 'mixed' in a mathematical theory are merely different (such as ordinal and closure), while in others they are opposites (as in the cases where he sees mixtures of finite and infinite mathematics).

2 Plato does not suffice

Lautman scattered references to Plato throughout his works; Heidegger, on the other hand, does not feature anywhere in his writing other than the discussion in *New research on the dialectical structure of mathematics* and implicitly in some brief remarks in the conclusion to *Essay on the notions of structure and existence* (Lautman, 2006, pp. 228–229).[13] We may therefore suppose that Lautman turned to Heidegger in order to solve a particular problem in his overall Platonism.[14] Moreover, the Heideggerian text that he refers to, *On the Essence of Ground*, is a meditation on the 'ontological difference' between the 'ontic' concepts employed in the sciences and the

[13]The sole exception is in a short piece of 1933 *Considérations sur la logique mathématique*. But here he discusses the use that the intuitionists made of phenomenology and makes no commitment of his own: "Les intuitionnistes se rattachent par là aux phénoménologues disciples de Husserl, Heidegger, et Oscar Becker" (Lautman, 2006, p. 43).

[14]Which is not to suggest that Lautman chose Heidegger arbitrarily, given his references to Plato; Heidegger prefaced *Being and Time* with a quotation from the *Sophist* (244a), and he devoted his lectures of 1924/25 to that same dialogue.

underlying 'ontological' concepts disclosed by phenomenology. The relation between dialectics and mathematics was clearly problematic for Lautman. On one hand, he was committed to his Platonist view that Ideas are somehow prior to the matter that they dominate, and which participates in them. In a talk given in 1937, Lautman claims that, "The reality inherent in mathematical theories is due to their participation in an ideal reality which dominates mathematics, but which cannot be known except through mathematics"[15]. He knew that the logical empiricist mainstream would regard his view as a mystification, "as obscure as the mystical beliefs of primitives in the participation of subjects in objects of which Mr. Lévy-Bruhl speaks".[16] He retorts that, on the contrary, empiricism (whether Aristotelian or Viennese) separates thought from experience and thus makes a mystery of the fact that we find nature mathematically intelligible. Moreover, a tautological view of mathematics separates the discovery of truth from the quest for reality (since tautologies do not require reference to any reality). Empiricism, he thought, deprives science of its spiritual dignity and value. Thus, it is scientifically and spiritually vital to insist on the reality of dialectical notions and the Ideas of their possible relations prior to their realisation in particular cases. On the other hand, notions only come into relations with each other when 'mixed' in particulars. Towards the end of the *Essay on the notions of Structure and Existence*, Lautman characterises Ideas of possible relations between notions as 'problems' or 'questions' and actual (realised) relations between notions as 'logical schemas':

> The logical schemas that we have described are not prior to their realisation at the heart of a theory; what is lacking from [...] the extra-mathematical intuition of the urgency of a logical problem is that it must have material to dominate, for the idea of possible relations to give birth to a scheme of real relations.[17]

Before the development of the mathematical theory that solves the problem, there is only "the experience of the urgency of problems".[18] However, this formulation makes it sound as if we are concerned with the psychology

[15]"La réalité inhérente aux théories mathématiques leur vient de ce qu'elles participent à une réalité idéale qui est dominatrice par rapport à la mathématique, mais qui n'est connaissable qu'à travers elle" (Lautman, 2006, pp. 67–68).

[16]"[...] aussi obscures que les croyances mystiques à la participation du sujet à l'objet chez les primitifs dont parle M. Lévy-Bruhl" (Lautman, 2006, p. 64).

[17]"Les schémas logiques que nous avons décrits ne sont pas antérieurs à leur réalisation au sein d'une théorie; il manque en effet à ce que nous appelons plus haut l'intuition extra-mathématique de l'urgence d'un probléme logique, une matière à dominer pour que l'idée de relations possibles donne naissance au schéma de relations véritables" (Lautman, 2006, p. 229).

[18]"Le seul élément *a priori* que nous concevions est donné dans l'expérience de cette urgence des problèmes[...]" (Lautman, 2006, p. 229).