

STRING MODULAR PHASES IN CALABI-YAU FAMILIES

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Dedicated to the memory of Max Kreuzer

Abstract:

We investigate the structure of singular Calabi-Yau varieties in moduli spaces that contain a Brieskorn-Pham point. Our main tool is a construction of families of deformed motives over the parameter space. We analyze these motives for general fibers and explicitly compute the L -series for singular fibers for several families. We find that the resulting motivic L -functions agree with the L -series of modular forms whose weight depends both on the rank of the motive and the degree of the degeneration of the variety. Surprisingly, these motivic L -functions are identical in several cases to L -series derived from weighted Fermat hypersurfaces. This shows that singular Calabi-Yau spaces of non-conifold type can admit a string worldsheet interpretation, much like rational theories, and that the corresponding irrational conformal field theories inherit information from the Gepner conformal field theory of the weighted Fermat fiber of the family. These results suggest that phase transitions via non-conifold configurations are physically plausible. In the case of severe degenerations we find a dimensional transmutation of the motives. This suggests further that singular configurations with non-conifold singularities may facilitate transitions between Calabi-Yau varieties of different dimensions.

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1 Introduction and outline

Previous work has provided support for the idea that the problem of deriving the geometry of spacetime in string theory from first principles can be approached via the concept of automorphic motives. The strategy of this framework is to relate automorphic forms derived from the geometry of spacetime to automorphic forms determined by the two-dimensional conformal field theory on the string worldsheet. It is possible to invert this process, and to construct the geometry of spacetime from the string theoretic automorphic forms. More precisely, it is possible to derive motivic pieces of the compactification variety from forms determined by the worldsheet theory. The combination of the resulting motivic building blocks then determines the global structure of the Calabi-Yau space. This approach thus leads to an explicit and computable realization of the concept of an emergent spacetime in string theory. The

strategy outlined above has been pursued in the recent past in the context of weighted Fermat hypersurfaces and Gepner models in a number of papers (see e.g. [1, 2, 3] and references therein).

Weighted Fermat hypersurfaces, or Brieskorn-Pham varieties, are special in that the underlying conformal field theory is rational. Rationality of the theory implies that the algebra and its representations are under precise control and it is possible to understand the relationship between the worldsheet theory and the geometry of Calabi-Yau manifolds in some detail. In the early years after the original construction of Gepner [4] the techniques applied to this end came from Landau-Ginzburg theories [5, 6, 7] and non-linear sigma models [8]. The arithmetic geometric approach pursued in [1, 2] in the context of weighted Fermat hypersurfaces and Gepner models relates the geometric modular forms derived from motives of the Calabi-Yau varieties to Hecke indefinite modular forms. These in turn are built from the string functions that appear in the partition function of the exactly solvable model. This method thereby establishes a close connection between modular forms of rather different origins.

Since the constructions of Gepner and Kazama-Suzuki two decades ago [4, 9], it has proven difficult to extend the detailed understanding obtained for rational theories to deformations of such theories. Much less, therefore, is known about the conformal field theories that correspond to families of Calabi-Yau varieties. Our purpose in this paper is to initiate a program to address this problem by extending the modularity methods used for K3 surfaces and Calabi-Yau threefolds of Brieskorn-Pham type in [1, 2] to the class of Calabi-Yau families which contain a Brieskorn-Pham fiber in their moduli space. The structure of such families is of the form

$$X_n^d(\psi_D) = \left\{ (z_0 : \cdots : z_{n+1}) \in \mathbb{P}_{(w_0, \dots, w_{n+1})} \mid P(z_i, \psi_D) = p_{\text{BP}}(z_i) - \sum_{J \in D} \psi_J z^J = 0 \right\} \quad (1)$$

where

$$p_{\text{BP}}(z_i) = \sum_{i=0}^{n+1} z_i^{d_i} \quad (2)$$

is the weighted Fermat polynomial of degree d , with integral $d_i = d/w_i$. The monomials $z^J = \prod_{i=0}^{n+1} z_i^{j_i}$, with $J = (j_0, \dots, j_{n+1})$ for $j_i \in \mathbb{N}$ such that $\sum_{i=0}^{n+1} j_i w_i = d$ define the deformations. We denote the set of all deformation vectors J that appear in the polynomial by $D \subset \mathbb{Z}^{n+2}$.

In the context of this framework we analyze two separate but ultimately related issues. First,

we generalize our previous results concerning the string theoretic interpretation of certain types of geometric modular forms from weighted Fermat hypersurfaces to families of Calabi-Yau varieties. Second, we use these generalized modularity results to investigate the conformal field theoretic structure of particular types of singular Calabi-Yau spaces.

The first step in this program involves the computation of geometric L -series defined via motives $M_\Omega(X(\psi_D))$ constructed from the families $X(\psi_D)$ defined by (1). Our construction of the motives of general families of n -dimensional weighted projective hypersurfaces involves the deformation of the Ω -motive defined at the Brieskorn-Pham point by the deformation monomials z^J , $J \in D$. For any vector J we define an action on the Ω -motive $M_\Omega(X(0))$ of the Brieskorn-Pham variety $X(0)$, which we denote here by $\rho(J)M_\Omega(X(0))$. The deformed motive of the family defined by D is then defined as the sum of all contributions spanned by the action of the deformation vectors on the Ω -motive of the Brieskorn-Pham fiber

$$M_\Omega(X(\psi_D)) = \bigoplus_{J \in D} \rho(J)M_\Omega(X(0)). \quad (3)$$

This construction will be made precise in Section 3.

For smooth fibers the motives $M_\Omega(X(\psi_D))$ defined by (3) have in general higher rank and therefore are not modular in terms the congruence subgroups of the modular group $SL(2, \mathbb{Z})$. The Langlands program suggests that such higher rank motives instead lead to automorphic forms associated to higher rank groups. We will see later in this paper that for singular fibers of the families of weighted projective hypersurfaces the motives (3) show interesting degeneration behavior. As a result, families of motives that are of high rank in the generic smooth fibers can become modular for singular Calabi-Yau fibers. This phenomenon can be used to investigate the modular structure of such singular configurations. In the past, most of the work on singular string compactifications has focused on conifolds, varieties with nodes. In particular, the D-brane analysis by Greene, Strominger and Morrison [10, 11] of the geometric transitions introduced in [12], and analyzed in detail in [13, 14, 15], led to a paradigmatic interpretation.

Conifold singularities are not the only type of degeneration that occur in moduli space, and the singularity types that we consider in the present case are of more severe type, characterized by Milnor numbers that are larger than one. For such more general singularities a D-brane interpretation is lacking, which raises the question whether degenerate Calabi-Yau varieties of

non-conifold type admit a physical interpretation. For Brieskorn-Pham varieties the underlying worldsheet theory is a rational conformal field theory, and previous work has shown that the modular forms derived for such spaces have a string theoretic interpretation. The generalization of this string modularity program away from rational theories has not been addressed previously. The second purpose of this paper is to investigate the conformal field theoretic properties of the singular fibers in families of Calabi-Yau varieties. This leads to a surprising result for the deformed motives. As mentioned above, for general fibers of the Calabi-Yau families the deformed motive is of high rank, and therefore not modular. The unexpected phenomenon encountered in this work is that it can happen that the motivic L -series of a non-Fermat type fiber in a family agrees with the motivic L -series of a weighted Fermat variety. We will call such pairs of models L -correlated. The existence of such a L -correlated model shows that the deformed fiber inherits information from the rational conformal field theory underlying the Brieskorn-Pham manifold. This result makes it plausible that singular configurations of such a type could serve as links, much like conifolds, between different moduli spaces of smooth configurations.

A second phenomenon we encounter in the present investigation is that of dimensional reduction of the motives at the singular fibers of the families. It happens that in some families of varieties the modular forms of the deformed motives change their weight at the singular fibers. The weight of these forms is lower than the weight of the automorphic forms expected for the smooth generic fiber of the family. Since the deformed Ω -motive probes the dimension of the ambient space, and is not induced by lower-dimensional subvarieties, this suggests that phase transitions between Calabi-Yau varieties possibly involve motives that effectively have different dimensions. Even in these extreme cases the modular forms that emerge contain information coming from the structure of the underlying worldsheet. We will call this phenomenon motivic dimensional transmutation.

We exemplify our strategy with a number of one- and two-dimensional families of the type

$$X_n^d(\psi_1, \psi_2) = \left\{ (z_0 : \cdots : z_{n+1}) \in \mathbb{P}_{(w_0, \dots, w_{n+1})} \mid p_{\text{BP}}(z_i) - 2\psi_1 z_{n-2}^a z_{n-1}^a - 2\psi_2 z_n^b z_{n+1}^b = 0 \right\}, \quad (4)$$

where we have simplified the notation for the deformation parameter for this particular case. For the special subclass of K3 surfaces these families extend the two-parameter family of quartic surfaces obtained for $d_i = 4 \forall i$, which was considered in depth in a different con-

text and with different methods by Wendland [16]. In Section 2 we describe the general motivic framework and in Section 3 we detail our construction of deformed Ω -motives and their L -functions. In Section 4 we compute these L -functions for families with up to two parameters of the type (4) for K3 surfaces and Calabi-Yau threefolds. In Section 5 we focus on singular fibers of non-conifold type and present some modularity results, and in Section 6 we consider the conformal field theoretic aspects of the resulting modular forms. In Section 7 we illustrate the phenomenon of motivic dimensional transmutation for K3 surfaces and Calabi-Yau threefolds, and we conclude in Section 8.

2 Zeta functions and Weil conjectures for varieties and motives

The Weil conjectures, proven by Dwork, Grothendieck and Deligne, give a characterization of the congruence zeta function in terms of the cohomology groups of the varieties, thereby turning a pure counting function into an object that contains topological information. In the past, conformal field theoretic applications of arithmetic geometry have focussed on smooth projective manifolds, or varieties in weighted projective spaces. In the present paper we extend these applications to the case of more general varieties, allowing for singular spaces. We therefore briefly describe the behavior of the underlying arithmetic tools in such a generalized framework.

2.1 Arithmetic of general varieties

A useful starting point for any discussion of motives is the cohomological interpretation of the zeta function, conjectured originally by Weil [17], and proven by Grothendieck [18]. This cohomological interpretation amounts to a factorization of this function which holds for any variety¹. Let X be a variety of complex dimension n over a finite field \mathbb{F}_q , where $q = p^r$ for primes p and $r \in \mathbb{N}$. The Artin zeta function over \mathbb{F}_q , defined as

$$Z(X/\mathbb{F}_q, t) = \exp \left(\sum_{r \geq 1} \frac{N_{r,q}(X)}{r} t^r \right), \quad (5)$$

¹A variety is defined here as a separated scheme of finite type.

factors as follows

$$Z(X/\mathbb{F}_q, t) = \frac{\prod_{i=0}^{n-1} \mathcal{P}_q^{2i+1}(X, t)}{\prod_{j=0}^n \mathcal{P}_q^{2j}(X, t)}, \quad (6)$$

where $\mathcal{P}_q^i(X, t)$ are polynomials whose degree is given by the Betti numbers $b^i(X) = \dim H^i(X)$ of the variety X

$$\deg \mathcal{P}_q^i(X, t) = b^i(X). \quad (7)$$

Consider the factorization of the polynomials

$$\mathcal{P}_p^i(X, t) = \prod_j (1 - \gamma_j^i(p)t). \quad (8)$$

The generalized Riemann hypothesis is a characterization of the complex numbers $\gamma_j^i(p)$. The precise form of this hypothesis depends on the detailed structure of the variety. For smooth projective manifolds it was shown by Deligne [19] that the numbers $\gamma_j^i(p)$ are algebraic and satisfy

$$|\gamma_j^i(p)| = p^{i/2}. \quad (9)$$

For general varieties which admit singularities a weaker constraint applies, as proven in [20]

$$|\gamma_j^i(p)| \leq p^{(i-\ell)/2}, \quad (10)$$

for some $\ell \geq 0$. Here ℓ depends on the type of singularity of the variety. We will describe several examples with varying ℓ in this paper.

2.2 Pure and mixed motives

The notion of a motive of a variety was formulated by Grothendieck as a means to identify the essential underlying geometric structure that supports the various cohomology theories of a variety. The goal is to isolate the fundamental irreducible building blocks of arbitrary varieties, analogous to the idea of fundamental particles in physics. In this picture, varieties are viewed as constructed out of fundamental ‘geometric particles’, with the same motives appearing in completely different manifolds, much like all atoms are built with a few particles. While the concept of motives in its most general formulation has remained somewhat elusive, in the present paper we build on the framework of Grothendieck’s effective motives with Tate twists.

These are defined via projectors γ in the ring of correspondences, which are multi-valued maps between varieties. A Grothendieck type motive is therefore defined as a triplet

$$M = (X, \gamma, m), \quad (11)$$

where $\gamma^2 = \gamma$, and $m \in \mathbb{Z}$ denotes the Tate twist. The special class of motives with $m = 0$ are called effective motives. Implicit in this definition is the choice of an equivalence relation that defines the correspondences.

The correspondence γ induces a projection on the cohomology, denoted by the same symbol, $\gamma H^*(X)$. This makes it possible to think of the motive M in terms of its cohomological representation $H(M)$ for various types of cohomology groups $H^*(X)$ associated to the variety, i.e. $H(M) \subset H^*(X)$. Pure motives are distinguished by the fact that there exists an integer w , the weight $\text{wt}(M)$ of the motive, such that the cohomological representative of the motive $M = (X, \gamma, m)$ is given by the projection γ acting on the cohomology of degree n , where $\text{wt} = n - 2m$. Hence we have

$$H(M) = \gamma H^n(X)(m). \quad (12)$$

It follows that the zeta function of an effective motive of weight w defines a polynomial

$$Z(M/\mathbb{F}_q, t) = \mathcal{P}_q(M, t)^{(-1)^{n+1}} \quad (13)$$

that divides $\mathcal{P}_q^n(X, t)$. The degree of this polynomial is determined by the rank of the motive, which can be defined in terms of $H(M)$ as

$$\text{rk } M = \dim H(M) \quad (14)$$

in $H^*(X)$. The L -function of a Grothendieck type motive can then be defined as the product over all primes with $t = p^{-s}$

$$L(M, s) = \prod_p \frac{1}{\mathcal{P}_p(M, p^{-s})}, \quad (15)$$

where the detailed structure of the polynomials $\mathcal{P}_p(M, p^{-s})$ depends on whether the primes are good or bad. More details about the concept of pure motives can be found in [2]. Physical applications of this notion in the context of transversal higher-dimensional hypersurfaces in weighted projective spaces are described in [1, 2, 3]. A more advanced treatment can be found in [21].

A general formulation of mixed motives is not known at present. It is expected that associated to each mixed motive M is an ascending weight grading $W_n M$, with $W_n M = 0$ for $n \ll 0$ and $W_n M = M$ for $n \gg 0$, such that the resulting quotients $\text{Gr}_n^w M = W_n M / W_{n-1} M$ are pure. A motive of weight w is pure if $\text{Gr}_n^w M = 0$ for $n \neq w$. The degree of the polynomials $\mathcal{P}_p(M, t)$ of mixed motives is no longer determined by the simple relation that describes pure motives.

3 The Ω -motive of Calabi-Yau families

To define a Grothendieck motive means to define a projection γ . In the present section we construct such γ for general families of Calabi-Yau hypersurfaces that contain a Brieskorn-Pham point in their moduli space. Our idea is to consider projections that are constructed from the deformations of the Ω -motives defined for Brieskorn-Pham type hypersurfaces in [1], and for more general varieties in [2]. We denote the Ω -motive of the weighted Fermat hypersurface $X(0)$ by $M_\Omega(X(0))$. To explain the construction of the motive of the family $X(\psi_D)$, with ψ_J for $J \in D$ denoting the vector of moduli parametrizing the different deformations, as in (3), it is necessary to construct an action ρ_J induced by the deformation z^J in the J -direction. The realization of this action depends on the particular representation of the motive $M_\Omega(X(0))$ for the Brieskorn-Pham fiber. In the following, we briefly review the specialization of the Ω -motive to Brieskorn-Pham fibers [2] and then define the action of the deformations on this motive.

With the ingredients introduced in the previous section one can define a universal type of motive characteristic for Calabi-Yau varieties as follows [1, 2].

Definition. *Let X be a Calabi-Yau manifold of complex dimension n , $n \in \mathbb{N}$. Define the number field K_X as the extension of \mathbb{Q} given by the algebraic numbers γ_j^n defined by the factorization of the polynomial $\mathcal{P}_p^n(t)$ associated to its intermediate cohomology, $K_X = \mathbb{Q}(\{\gamma_i^n(p)\})$. The cohomological representation of the Ω -motive $M_\Omega(X)$ is defined as the orbit of the holomorphic n -form of X with respect to the Galois group $\text{Gal}(K_X/\mathbb{Q})$.*

The notion of the Ω -motive can also be applied to the more general class of manifolds of special Fano type, which includes Calabi-Yau varieties as a special case. These spaces were

originally introduced in the context of mirror symmetry in [22, 23, 24], and have been discussed in the context of arithmetic mirror symmetry. It was shown in ref. [3] that the modular forms of certain rigid Calabi-Yau spaces agree with the modular forms of their mirrors of special Fano type.

Consider now the general class of Calabi-Yau hypersurface families defined in (1). In the context of these varieties the Ω -motive can be described concretely as the motive that is obtained from the action of the moduli on the Ω -motive of the Brieskorn-Pham fiber of the family. In this framework an explicit formulation of the deformed motive can be given as follows. Define for any weighted hypersurface family $X_n^d(\psi_D)$ of dimension n and degree d embedded in weighted projective space $\mathbb{P}_{(w_0, \dots, w_{n+1})}$ the set of vectors

$$\mathcal{U} = \left\{ u \in \mathbb{Z}_{\geq 0}^{n+2} \mid 0 \leq u_i \leq d_i - 1 \text{ and } d \mid \sum_i w_i u_i \right\}, \quad (16)$$

where d, w_i, d_i are as in eq. (1). To each vector $u \in \mathcal{U}$ with $u_i \neq 0, \forall i$ we associate a monomial via $z^{u-1} = \prod_i z_i^{u_i-1}$. These monomials parametrize (part of) the cohomology of the fibers of the family. The vector $(1, 1, \dots, 1)$, denoted by u_Ω in the following, is distinguished by the fact that it represents the holomorphic n -form on the variety X_n .

In terms of the vector u_Ω defined above, the Brieskorn-Pham type Ω -motive can be parametrized by the Galois orbit

$$\mathcal{O}_{\Omega(0)} = \bigoplus_{g \in \text{Gal}(K_X/\mathbb{Q})} \rho(g) u_\Omega, \quad (17)$$

where ρ is the representation of $\text{Gal}(K_X/\mathbb{Q})$ on \mathcal{U} . Since K_X is a cyclotomic field, the elements $g_n \in \text{Gal}(K_X/\mathbb{Q})$ are defined on the generator ξ of the field K_X as $g_n \xi = \xi^n$. The action $\rho(g_n)$ on u then is defined as $\rho(g_n)u = nu \pmod{d}$.

We construct the family of motives over the moduli space given by (ψ_D) via the orbit of u -vectors that act on the BP type motive. A general combination of monomial deformations is determined by a subset of u -vectors, denoted in eq. (1) by $J \in D \subset \mathcal{U}$. For any family of varieties defined by D the deformed motive $M(X(\psi_D))$ is defined by the orbit that is obtained by considering

$$\mathcal{O}_{\Omega(\psi_D)} := \bigoplus_{J \in D} r(J) \mathcal{O}_{\Omega(0)}, \quad (18)$$

where the representation $r(J)$ on the u -vectors defining the Ω -motive via the orbit $\mathcal{O}_{\Omega(0)}$ is the additive action. The orbit $\mathcal{O}_{\Omega(\psi_D)}$ is to be viewed as a cohomological realization of the motive $M_{\Omega}(X(\psi_D))$ introduced in eq. (3).

In order to determine the L -functions $L(M_{\Omega}(X(\psi_D), s)$ of the motives $M_{\Omega}(X(\psi_D))$ generated by the orbits $\mathcal{O}_{\Omega(\psi_D)}$, it is useful to first consider the cardinalities of the varieties in terms of Gauss sums. This is what we turn to in the next section.

4 Motivic L -functions for weighted hypersurface families

In this section we apply our general construction to the special case of families of Calabi-Yau hypersurfaces in weighted projective spaces.

4.1 Cardinalities

For a complete motivic analysis of a variety, it is necessary to have the complete cardinality structure of the variety under control. There are several ways to compute this, using either p -adic or complex methods. Both lend themselves equally well to explicit computations. We find that the complex method is more transparent as far as the motivic structure of the varieties is concerned, while p -adic methods have traditionally been pervasive in arithmetic geometry, ever since Dwork's proof of the rationality of the zeta function (recent applications in physics can be found in [25, 26, 27, 28, 29]). For Brieskorn-Pham varieties complex techniques originated with Gauss in the context of the cubic elliptic curve, and the special case of the canonical family of Fermat hypersurfaces was considered by Koblitz [30].

The main ingredients in the computation of the cardinalities are Gauss sums $G_{n,p}$ that are determined by two characters, the multiplicative character χ_p and the additive character Ψ_p , both associated to the finite field \mathbb{F}_p . The multiplicative character is given by the map

$$\chi_p : \mathbb{F}_p^{\times} \longrightarrow \mu_{p-1}, \quad (19)$$

defined by $\chi_p(u) = \xi_{p-1}^m$, where $\xi_n = e^{2\pi i/n}$ and m is determined by the generator $g \in \mathbb{F}_p^{\times}$ via

$u = g^m$. The additive character

$$\Psi_p : \mathbb{F}_p \longrightarrow \mu_p, \quad (20)$$

can be defined by $\Psi_p(u) = \xi_p^u$.

The two characters χ_p and Ψ_p can be lifted to characters on the degree r extension \mathbb{F}_q where $q = p^r$ for primes p and $r \in \mathbb{N}$, via the norm and the trace map, respectively. The norm map is defined for \mathbb{F}_q as

$$N_r : \mathbb{F}_q \longrightarrow \mathbb{F}_p \quad (21)$$

where

$$N_r(v) = v \cdot v^p \cdots v^{p^{r-1}}. \quad (22)$$

The trace map is given by

$$\mathrm{Tr}_r : \mathbb{F}_q \longrightarrow \mathbb{F}_p \quad (23)$$

with

$$\mathrm{Tr}_r(v) = v + v^p + v^{p^2} + \cdots + v^{p^{r-1}}. \quad (24)$$

The lifts of the characters χ_p and Ψ_p are then defined as the compositions

$$\chi_q(v) = \chi_p \circ N_r, \quad \Psi_q(v) = \Psi_p \circ \mathrm{Tr}_r. \quad (25)$$

The Gauss sum $G_{n,q}$ is defined in terms of the characters χ_q, Ψ_q as

$$G_{n,q} = \sum_{v \in \mathbb{F}_q^\times} \chi_q(v)^n \Psi_q(v). \quad (26)$$

In the context of the p -adic approach, the character Ψ_p is replaced by the Dwork character, and the multiplicative character χ_p is replaced by the Teichmüller character (see e.g. [25]).

The additive character satisfies the vanishing relation

$$\sum_{x \in \mathbb{F}_q} \Psi_q(yx) = \begin{cases} q & \text{if } y = 0 \\ 0 & \text{if } y \in \mathbb{F}_q^\times \end{cases} \quad (27)$$

and therefore lends itself to the computation of cardinalities of polynomial varieties.

An efficient strategy for deriving the cardinalities of projective varieties and their motives is to first consider the multiplicative cardinalities of the affine varieties $X(\psi_D)_{\mathrm{aff}}$, denoted here by

$N_q^\times(X(\psi_D)_{\text{aff}})$. The projective cardinalities can be determined by first computing the affine cardinalities $N_q(X(\psi_D)_{\text{aff}})$ via an iterative procedure that takes into account the multiplicative affine cardinalities of lower-dimensional strata, and by implementing the projective equivalence relation. To this effect, we define the subvarieties $X_{n-j}(\psi_D)$ of $X_n(\psi_D)$ via the hyperplane intersections

$$H_{i_1, \dots, i_j} = \{x_{i_1} = 0\} \cap \dots \cap \{x_{i_j} = 0\} \cap \mathbb{P}_{(w_0, \dots, w_{n+1})} \quad (28)$$

as

$$X_{n-j}(\psi_D) = X_n(\psi_D) \cap H_{i_1, \dots, i_j}. \quad (29)$$

Collecting all the contributions of the multiplicative affine cardinalities of the subvarieties (29) leads to the affine cardinalities

$$N_q(X(\psi_D)_{\text{aff}}) = \sum_{j=0}^{n-1} N_q^\times(X_{n-j}(\psi_D)_{\text{aff}}). \quad (30)$$

Projectivizing the affine cardinalities then leads to cardinalities of the projective varieties

$$N_q(X_n(\psi_D)) = \frac{1}{q-1} (N_q(X_n(\psi_D)_{\text{aff}}) - 1). \quad (31)$$

The motives that we consider are those of the projective varieties.

The starting point for the derivation of the affine multiplicative cardinalities is their representation in terms of the additive character

$$N_q^\times(X_n^d(\psi_D)_{\text{aff}}) = \frac{1}{q} \sum_{y \in \mathbb{F}_q} \sum_{z_i \in \mathbb{F}_q^\times} \Psi_q(yP(z_i, \psi_D)), \quad (32)$$

where $P(z_i, \psi_D)$ is the polynomial in eq. (1). In order to perform the \mathbb{F}_q -sums, it is useful to factorize $P(z_i, \psi_D)$ as much as possible. This can be achieved by trading the additive character for the multiplicative character by inverting the Gauss sums

$$\Psi_q(x) = \frac{1}{q-1} \sum_{m=0}^{q-2} G_{-m, q} \chi_q(x)^m. \quad (33)$$

The final input needed in the computation is the relation

$$\sum_{x \in \mathbb{F}_q^\times} \chi_q(x)^m = \begin{cases} (q-1) & \text{if } (q-1) | m \\ 0 & \text{otherwise} \end{cases}. \quad (34)$$

With these ingredients we obtain explicit results for the cardinalities of families of weighted projective hypersurfaces $X_n \subset \mathbb{P}_{(w_0, \dots, w_{n+1})}$ as well as the motives defined above. In what follows, we introduce for q with $d|(q-1)$ the integer $k = (q-1)/d$. For families of varieties the cardinality formulae depend on the detailed form of the deformations. In the following, we consider certain classes of one- and two-parameter families of weighted hypersurfaces in arbitrary dimensions such that the family contains a Brieskorn-Pham type fiber. It is useful to introduce the following Gauss sum products for the zero-, one-, and two-parameter families

$$\begin{aligned}
\mathbb{G}_q^n(u) &= \prod_{i=0}^{n+1} G_{-w_i u_i k, q} \\
\mathbb{G}_q^n(u, \psi) &= \mathbb{G}_q^{n-2}(\underline{u}) \sum_{\ell=0}^{\frac{d}{2}k-1} G_{2\ell, q} \chi_q(-2\psi)^{-2\ell} \left(\prod_{j=n}^{n+1} G_{-(\ell+w_j u_j k), q} \right) \\
\mathbb{G}_q^n(u, \psi_1, \psi_2) &= \mathbb{G}_q^{n-3}(\underline{u}) \sum_{\ell_1, \ell_2=0}^{\frac{d}{2}k-1} \mathbb{G}_{2\ell_1, q} \mathbb{G}_{2\ell_2, q} \chi_q(-2\psi_1)^{-2\ell_1} \chi_q(-2\psi_2)^{-2\ell_2} \\
&\quad \cdot \left(\prod_{i=n-2}^{n-1} G_{-(\ell_1+w_i u_i k), q} \right) \left(\prod_{j=n}^{n+1} G_{-(\ell_2+w_j u_j k), q} \right). \tag{35}
\end{aligned}$$

Here \underline{u} denotes the truncated u -vector of the reduced Gauss sum product. For the one-parameter family $\underline{u} = (u_0, \dots, u_n)$, while for the two-parameter case $\underline{u} = (u_0, \dots, u_{n-1})$.

We find that the cardinalities of our varieties can be written in terms of the Gauss sum polynomials of eq. (35) in the following expressions, amenable to explicit computations.

Result 1. *The multiplicative affine cardinalities of Brieskorn-Pham type hypersurfaces of dimension n and degree d for prime powers q such that $d|(q-1)$ are given by*

$$N_q^\times(X_n(0)_{\text{aff}}) = \frac{(q-1)^{n+2}}{q} + \frac{(q-1)}{q} \sum_{u \in \mathcal{U}} \mathbb{G}_q^n(u), \tag{36}$$

For the multiplicative affine cardinalities of the one- and two-parameter families the remaining Gauss sum products of (35) enter.

Result 2. *Define the one-parameter family $X_n^d(\psi)$ by*

$$X_n^d(\psi) = \left\{ (z_0 : \dots : z_{n+1}) \in \mathbb{P}_{(w_0, \dots, w_{n+1})} \mid p_{\text{BP}}(z_i) - 2\psi z_n^a z_{n+1}^a = 0 \right\}. \tag{37}$$

The multiplicative affine cardinalities for prime powers q such that $d|(q-1)$ are then given by

$$N_q^\times(X_n^d(\psi)_{\text{aff}}) = \frac{(q-1)^{n+2}}{q} + \frac{1}{q} \sum_{u \in \mathcal{U}} \mathbb{G}_q^n(u, \psi). \quad (38)$$

For the two-parameter families

$$X_n^d(\psi_1, \psi_2) = \left\{ (z_0 : \cdots : z_{n+1}) \in \mathbb{P}_{(w_0, \dots, w_{n+1})} \mid p_{\text{BP}}(z_i) - 2\psi_1 z_{n-2}^a z_{n-1}^a - 2\psi_2 z_n^b z_{n+1}^b = 0 \right\} \quad (39)$$

the cardinalities are given by

$$N_q^\times(X_n^d(\psi_1, \psi_2)_{\text{aff}}) = \frac{(q-1)^{n+2}}{q} + \frac{1}{q(q-1)} \sum_{u \in \mathcal{U}} \mathbb{G}_q^n(u, \psi_1, \psi_2). \quad (40)$$

Here the Gauss sum products $\mathbb{G}_q^m(u, \psi)$ and $\mathbb{G}_q^m(u, \psi_1, \psi_2)$ are defined as in (35).

The cardinalities for other primes can be determined with the same techniques. In general, more primes than the ones considered in the second result are necessary to determine the modular forms uniquely, not only up to twists. For the hypersurfaces considered in this paper, the primes of the type $p \equiv 1 \pmod{d/2}$ are sufficient to uniquely determine the motivic modular form.

4.2 L -functions of the motives

By projectivizing the cardinality results obtained above we define the contributions of the Ω -motive to the cardinalities for the Brieskorn-Pham varieties as

$$M_\Omega(X_n^d(0)) : a_p = \frac{(-1)^{n-1}}{p} \sum_{u \in \mathcal{O}_{\Omega(0)}} \mathbb{G}_p^n(u). \quad (41)$$

For the fibers of a family with several parameters we denote by $|D|$ the dimension of the family. For deformed motives that receive no contributions from lower dimensional strata in the transition from multiplicative affine to projective cardinalities, we define the motivic cardinalities as

$$M_\Omega(X_n^d(\psi_D)) : a_p(\psi_D) = \frac{(-1)^{n-1}}{p(p-1)^{|D|}} \sum_{u \in \mathcal{O}_{\Omega(\psi_D)}} \mathbb{G}_p^n(u, \psi_D). \quad (42)$$

With the convention that $\mathbb{G}_p(u, 0) = \mathbb{G}_p(u)$, for $|D| = 0$ (the case of no deformation), this expression formally reduces to (41) for Brieskorn-Pham motives.

If the zeta function polynomials associated to the motive do not factorize, the coefficients a_p computed in this way determine the coefficients of the L -function of the Ω -motive

$$L_\Omega(X(\psi_D), s) = L(M_\Omega(X(\psi_D)), s) = \sum_n \frac{a_n}{n^s}. \quad (43)$$

In the case the zeta function factorizes corresponding modifications have to be implemented, depending on the type of factorization.

In this paper we are mainly interested in the highest weight motives because they are most characteristic of the variety. The strategy described above for computing the L -function of these motives, however, applies to all motives defined by our deformation construction. The basic structure is that the intermediate cohomology group $H^n(X_n)$ of the variety X_n decomposes into orbits \mathcal{O}_i , each of which defines a motive M_i . The L -functions of these motives M_i then lead to lower weight modular forms via Tate twists.

In the remainder of this paper, we consider several examples of K3 and threefold families and some of their physical implications.

5 Modular phases of CY varieties

In this section we discuss in detail several examples provided by fibers of two 1-dimensional K3 families and two 2-dimensional families of Calabi-Yau threefolds. Furthermore, we list results for additional K3 surfaces and threefolds in Tables 2 and 3. Our discussion involves a number of different modular forms of different weights and levels, hence it is useful to collect these objects for later reference in Table 1¹. Some of these modular forms can be expressed in closed form in terms of the Dedekind eta function and the Eisenstein series. Denoting the Eisenstein field by $K_E = \mathbb{Q}(\sqrt{-3})$, these function are given as

$$\eta(q) = q^{1/24} \prod_{i \geq 1} (1 - q^i), \quad \vartheta(q) = \sum_{z \in \mathcal{O}_{K_E}} q^{Nz}, \quad (44)$$

¹Resources for modular forms are William Stein's websites, as well as the tables of Meyer [31] and Schütt [32].

respectively, where $q = e^{2\pi i\tau}$, with τ in the upper half plane, and Nz denotes the norm of the algebraic integer $z \in \mathcal{O}_{K_E}$.

Modular form	q -Expansion or closed form
$f_{2,27}(q) = \eta^2(q^3)\eta^2(q^9)$	$= q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} - 5q^{25} + 2q^{28} - 4q^{31} + 11q^{37} + \dots$
$f_{2,32}(q) = \eta^2(q^4)\eta^2(q^8)$	$= q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} - q^{25} - 10q^{29} - 2q^{37} + 10q^{41} + \dots$
$f_{3,16}(q) = \eta^6(q^4)$	$= q - 6q^5 + 9q^9 + 10q^{13} - 30q^{17} + 11q^{25} + 42q^{29} - 70q^{37} + \dots$
$f_{3,27}(q) = \eta^2(q^3)\eta^2(q^9)\vartheta(q^3)$	$= q + 4q^4 - 13q^7 - q^{13} + 16q^{16} + 11q^{19} + 25q^{25} - 52q^{28} - 46q^{31} + \dots$
$f_{3,108}(q)$	$= q + 11q^7 + 23q^{13} - 37q^{19} - 46q^{31} - 73q^{37} - 22q^{43} + 47q^{61} + \dots$
$f_{4,9}(q) = \eta^8(q)$	$= q - 8q^4 + 20q^7 - 70q^{13} + 64q^{16} + 56q^{19} - 125q^{25} - 160q^{28} + \dots$
$f_{4,32}(q)$	$= q + 22q^5 - 27q^9 - 18q^{13} - 94q^{17} + 359q^{25} - 130q^{29} + 214q^{37} + \dots$
$f_{4,108A}(q)$	$= q - 37q^7 - 19q^{13} - 163q^{19} - 125q^{25} + 308q^{31} + 323q^{37} + \dots$
$f_{4,108B}(q)$	$= q + 17q^7 + 89q^{13} + 107q^{19} - 125q^{25} + 308q^{31} - 433q^{37} + \dots$

Table 1. Motivic modular form factors $f_{w,N}(q) \in S_w(\Gamma_1(N))$ of weight w and level N .

Some of the motivic forms identified below are twists of the forms listed in Table 1, involving the Dirichlet characters defined by the Legendre symbol, denoted here by

$$\chi_n(p) = \left(\frac{n}{p} \right) = \left\{ \begin{array}{ll} 1 & \text{if } x^2 \equiv n \text{ for } x \in \mathbb{F}_p \\ -1 & \text{otherwise} \end{array} \right\}. \quad (45)$$

The level of the twisted form $f \otimes \chi$ depends on the level N of f , the conductor C_ϵ of the character ϵ of $f \in S_w(\Gamma_0(N), \epsilon)$, and the conductor C_χ of χ . With $C_\epsilon | N$ the level is given by

$$N(f \otimes \chi) = \text{lcm}\{N, C_\epsilon C_\chi, C_\chi^2\}. \quad (46)$$

The type of families we discuss in more detail are deformations of the Brieskorn-Pham hypersurfaces considered in [1, 2]. Specifically, for K3 surfaces, we consider the families defined by

$$\begin{aligned} X_2^{6B}(\psi) &= \left\{ z \in \mathbb{P}_{(2,2,1,1)} \mid z_0^3 + z_1^3 + z_2^6 + z_3^6 - 2\psi z_2^3 z_3^3 = 0 \right\} \\ X_2^8(\psi) &= \left\{ z \in \mathbb{P}_{(4,2,1,1)} \mid z_0^4 + z_1^4 + \sum_{i=2}^3 z_i^8 - 2\psi z_2^4 z_3^4 = 0 \right\}, \end{aligned} \quad (47)$$

while for Calabi-Yau threefolds we analyze in more detail the two-parameter families

$$\begin{aligned} X_3^6(\psi_1, \psi_2) &= \left\{ z \in \mathbb{P}_{(2,1,1,1,1)} \mid z_0^3 + \sum_{i=1}^4 z_i^6 - 2\psi_1 z_1^3 z_2^3 - 2\psi_2 z_3^3 z_4^3 = 0 \right\} \\ X_3^8(\psi_1, \psi_2) &= \left\{ z \in \mathbb{P}_{(4,1,1,1,1)} \mid z_0^2 + \sum_{i=1}^4 z_i^8 - 2\psi_1 z_1^4 z_2^4 - 2\psi_2 z_3^4 z_4^4 = 0 \right\}. \end{aligned} \quad (48)$$

The Ω -motives of the Brieskorn-Pham fibers at the origin of the families $X_2^{6B}(\psi)$ and $X_3^6(\psi_1, \psi_2)$ were shown in [1, 2] to be modular of weight 3 and 4, and levels $N = 27$ and $N = 108$, respectively. Modularity of the Brieskorn-Pham L -series follows from the fact that they are of complex multiplication type, i.e. they are purely number theoretic objects determined by Hecke characters Ψ_H . Such Hecke type L -series have been shown to be modular in Hecke's work [33], hence it follows that the motives that give rise to these characters are modular. The Hecke representation of the motivic modular forms of $X_2^{6B}(0)$ and $X_3^6(0)$ allows us to express the resulting modular forms in terms of Hecke indefinite modular forms that arise in the underlying conformal field theory [1, 2].

The rank of the motives of the smooth generic fibers of the above families $X_2^{6B}(\psi)$ and $X_3^6(\psi_1, \psi_2)$ varies between four and eight, indicating that the degeneration of the zeta functions depends strongly on the type of singularity of the configuration.

5.1 K3 surfaces

In this subsection we discuss two one-dimensional families in detail and summarize further results in Table 2.

$$\underline{X_2^{6B}(\psi) \subset \mathbb{P}_{(2,2,1,1)}}$$

Our first example is the family of elliptic K3 surfaces $X_2^{6B}(\psi)$ of eq. (47). The extremal Brieskorn-Pham fiber at $\psi = 0$ has been analyzed in detail in [1], in combination with the two other extremal K3 surfaces of Brieskorn-Pham type, $X_2^4(0) \subset \mathbb{P}_3$ and $X_2^{6A}(0) \subset \mathbb{P}_{(3,1,1,1)}$. To apply our construction, consider first the Ω -motive $M(X_2^{6B}(0))$ of the Brieskorn-Pham fiber in this family $X_2^{6B}(\psi)$. This motive is given by the Galois orbit of the holomorphic 2-form, as defined in (17), and can be represented by the u -vector orbit as

$$M_\Omega(X_2^{6B}(0)) : (1, 1, 1, 1) \oplus (2, 2, 5, 5). \quad (49)$$

The motive of the family $\psi \neq 0$ is obtained as the orbit generated by the deformation vector $J = (0, 0, 3, 3)$ acting on the Ω -motive $M(X_2^{6B}(0))$, as defined in (18), leading to

$$M_\Omega(X_2^{6B}(\psi)) : (1, 1, 1, 1) \oplus (1, 1, 4, 4) \oplus (2, 2, 5, 5) \oplus (2, 2, 2, 2). \quad (50)$$

For the L -series of the deformation motive $M_\Omega(X_2^{6B}(1))$ we obtain from eq. (42)

$$L_\Omega(X_2^{6B}(1), s) \doteq 1 - \frac{13}{7^s} - \frac{1}{13^s} + \frac{11}{19^s} - \frac{46}{31^s} + \frac{47}{37^s} + \dots \quad (51)$$

Here we ignore the bad primes at $p = 2, 3$.

Surprisingly, this L -function, given by the motive of a model that describes the deformation of a Brieskorn-Pham variety, is identical to the L -series of a different Brieskorn-Pham type K3 manifold, the surface $X_2^{6A}(0) \subset \mathbb{P}_{(3,1,1,1)}$ of the one-parameter family $X_2^{6A}(\psi)$. The L -series $L_\Omega(X_2^{6A}(0), s)$ has been computed previously in [1] via Jacobi sums, but also can be obtained by using the motivic cardinalities (41) in combination with the Gauss sums in eq. (35). These computations lead to the identity

$$L_\Omega(X_2^{6B}(1), s) = L_\Omega(X_2^{6A}(0), s). \quad (52)$$

This fact has a number of immediate consequences. First, the L -function of the deformed motive is a Hecke L -series because the L -series of the Brieskorn-Pham hypersurface is determined by the Hecke character associated to the Eisenstein field $K_E = \mathbb{Q}(\sqrt{-3})$, defined by $\psi_{6A}(\mathfrak{p}) = \mathfrak{p}$ for $\alpha_{\mathfrak{p}} \equiv 1 \pmod{3}$. It therefore follows from the work of Hecke that the corresponding q -series is modular. The result is the modular form $f_{3,27}(q)$ of weight 3 and level $N = 27$, listed in Table 1,

$$f_\Omega(X_2^{6A}(0), q) = \eta^2(q^3)\eta^2(q^9)\vartheta(q^3). \quad (53)$$

Here $\eta(q)$ and $\vartheta(q)$ are as defined in (44).

The relation between the deformed singular fiber and the weighted Fermat fiber thus immediately implies that the deformed Ω -motive of $X_2^{6B}(1)$ is modular, with the modular form given by that of the Brieskorn-Pham K3 surface

$$f_\Omega(X_2^{6B}(1), q) = f_\Omega(X_2^{6A}(0), q) \in S_3(\Gamma_1(27)). \quad (54)$$

This result also shows that the string model of the underlying deformed singular conformal field theory can lead to precisely the same structure as an exactly solvable model corresponding

to a smooth Brieskorn-Pham hypersurface, i.e. a rational theory. We will expand on this in Section 6, when we discuss the worldsheet interpretation of this result.

The fact that this K3 surface is elliptic suggests that it should be possible to identify modular forms of weight two in the singular fiber of the family $X_2^{6B}(\psi)$. The generic elliptic fiber in this family is a cubic curve embedded in the projective plane \mathbb{P}_2 . The elliptic motive in the family is given by the orbit

$$M_{\text{ell}}(X_2^{6B}(1)) : (1, 1, 2, 0) \oplus (1, 1, 5, 3) \oplus (2, 2, 4, 0) \oplus (2, 2, 1, 3), \quad (55)$$

and the associated L -function is given by

$$L(M_{\text{ell}}(X_2^{6A}(1)), s) \doteq 1 - \frac{1}{7^s} + \frac{5}{13^s} - \frac{7}{19^s} - \frac{4}{31^s} + \frac{11}{37^s} + \dots \quad (56)$$

This leads to the unique modular form of weight two and level 27, given by $f_{2,27}(q) = \eta^2(q^3)\eta^2(q^9) \in S_2(\Gamma_0(27))$. In a string theoretic context this form was discussed in [34], where it was shown that this worldsheet modular form matches that of the elliptic Fermat curve

$$E^3 = \left\{ (z_0 : z_1 : z_2) \in \mathbb{P}_2 \mid z_0^3 + z_1^3 + z_2^3 = 0 \right\}. \quad (57)$$

$$\underline{X_2^8(\psi) \subset \mathbb{P}_{(4,2,1,1)}}$$

The family $X_2^8(\psi)$ of octic K3 surfaces is elliptically fibered with generic fibers that are smooth quartic curves in the weighted projective plane $\mathbb{P}_{(2,1,1)}$. The Ω -motive at the Brieskorn-Pham point $\psi = 0$ is of rank four, and can be represented as the Galois orbit

$$M_{\Omega}(X_2^8(0)) : (1, 1, 1, 1) \oplus (1, 3, 3, 3) \oplus (1, 1, 5, 5) \oplus (1, 3, 7, 7). \quad (58)$$

The deformed motive determined by the vector $J = (0, 0, 4, 4)$ does not change this motive, hence we have in this particular case

$$M_{\Omega}(X_2^8(\psi)) = M_{\Omega}(X_2^8(0))$$

in the representation (3) and (18).

For the L -series of the motive $M_{\Omega}(X_2^8(1))$ we find with the one-parameter formula of eq. (35)

$$L_{\Omega}(X_2^8(1), s) \doteq 1 + \frac{6}{5^s} - \frac{10}{13^s} - \frac{30}{17^s} - \frac{42}{29^s} + \frac{70}{37^s} + \frac{18}{41^s} + \dots \quad (59)$$

This motivic L -series is also identical to the twist of the L -function of a Fermat hypersurface, in this case the quartic K3 surface $X_2^4(0) \subset \mathbb{P}_3$, defined by

$$X_2^4(0) = \left\{ (z_0 : \cdots : z_3) \in \mathbb{P}_3 \mid \sum_{i=0}^3 z_i^4 = 0 \right\}. \quad (60)$$

The twist character is given by the quadratic character χ_2 associated to the Gauss field $K_G = \mathbb{Q}(\sqrt{-1})$

$$L_\Omega(X_2^8(1), s) = L_\Omega(X_2^4(0), s) \otimes \chi_2. \quad (61)$$

The L -series of the quartic Fermat surface $X_2^4(0)$ is again a number theoretic object, determined by a Hecke character Ψ_H associated to the Gauss field K_G

$$L_\Omega(X_2^4(0), s) = L(\Psi_H^2, s). \quad (62)$$

Hence it follows again from Hecke's theory that this L -series is modular and that the modular form of the octic K3 surface is of CM-type, given by

$$f_\Omega(X_2^8(1), q) = \eta^6(q^4) \otimes \chi_2 \in S_3(\Gamma_1(64)), \quad (63)$$

with $\eta^6(q^4) \in S_3(\Gamma_0(16), \chi_{-1})$.

The string theoretic structure of the modular motive of the quartic K3 surface $X_2^4(0)$ was discussed in [1]. The underlying rational conformal field theory of $X_2^4(0)$ leads to Hecke indefinite modular forms $\Theta_{\ell, m}^k(\tau)$ with $k = 2$, and the theta series $\Theta_{1,1}^2(q) = \eta(q)\eta(q^2)$ completely determines the weight three modular form $\eta^6(q)$ via the symmetric square. Hence its twisted relative $f_\Omega(X_2^8(1), q)$ inherits the same rational conformal field theoretic structure, modified only by the appearance of the Dirichlet character.

The elliptic fibration structure of this K3 surface implies that there is a lower weight motive that provides a further modular object in this model. The elliptic motive is given by

$$M_{\text{ell}}(X_2^8(1)) : (1, 1, 2, 0) \oplus (1, 1, 6, 4) \oplus (1, 3, 6, 0) \oplus (1, 3, 2, 4), \quad (64)$$

which leads to the L -series

$$L(M_{\text{ell}}(X_2^8(1), s) = L(f_{2,32}, s) \otimes \chi_2, \quad (65)$$

where $f_{2,32}$ is the modular form of weight $w = 2$ and level $N = 32$ listed in Table 1. The modular form $f_{2,32} \otimes \chi_2$ is an element of $S_2(\Gamma_0(64))$. Its L -series is known to arise from the elliptic curve

$$E^4 = \left\{ (z_0 : z_1 : z_2) \in \mathbb{P}_{(1,1,2)} \mid z_0^4 + z_1^4 + z_2^2 = 0 \right\}, \quad (66)$$

a curve in the configuration of curves that appear in the elliptic fibration [36].

More K3 results:

Other K3 surfaces can be analyzed in the same way. In Table 2 we summarize the discussion so far and add further results. The K3 family $X_2^{12}(\psi) \subset \mathbb{P}_{(6,4,1,1)}$ is also an elliptic fibration, with the generic fiber given by degree six elliptic curves in the weighted projective plane $\mathbb{P}_{(1,2,3)}$. The L -series $L_\Omega(X_2^{12}(1), s)$ of the singular fiber of the family is again a Hecke L -series associated to a Hecke character, and is therefore modular. The resulting modular form admits complex multiplication and leads to a twist of the level $N = 27$ modular form $f_{3,27}(q)$ of weight 3 listed in Table 1. This form is thus again a symmetric square of a weight two modular form, this time associated to the Brieskorn-Pham elliptic curve embedded in $\mathbb{P}_{(1,2,3)}$. The structure of this K3 surface is therefore analogous to that of the degree six surface $X_2^6(1)$ considered in detail above, with the difference that the generic elliptic fiber is now of degree six, not of degree three.

K3 surface	Modular form
$X_2^{6A}(1) \subset \mathbb{P}_{(3,1,1,1)}$	$f_{3,108} \otimes \chi_3 \in S_3(\Gamma_1(432))$
$X_2^{6B}(1) \subset \mathbb{P}_{(2,2,1,1)}$	$f_{3,27} \in S_3(\Gamma_1(27))$
$X_2^8(1) \subset \mathbb{P}_{(4,2,1,1)}$	$f_{3,16} \otimes \chi_2 \in S_3(\Gamma_1(64))$
$X_2^{12}(1) \subset \mathbb{P}_{(6,4,1,1)}$	$f_{3,27} \otimes \chi_3 \in S_3(\Gamma_1(432))$

Table 2. *Modular fibers in 1-parameter K3 families.*

5.2 One and two-dimensional families of Calabi-Yau threefolds

Consider two-dimensional Calabi-Yau threefold families of the type

$$X_3^d(\psi_1, \psi_2) = \left\{ (z_0 : \cdots : z_4) \in \mathbb{P}_{(w_0, \dots, w_4)} \mid \sum_{i=0}^2 z_i^{d_i} - 2\psi_1 z_1^a z_2^a - 2\psi_2 z_2^b z_3^b = 0 \right\}, \quad (67)$$

with $d = 3, 4$, $a = d/2w_1, b = d/2w_3$, and $(w_0, \dots, w_4) \in \{(2, 1, 1, 1, 1), (4, 1, 1, 1, 1)\}$. The rank of the motive for the generic fibers is eight in both cases, but for the singular fibers at $(\psi_1, \psi_2) = (1, 1)$ the L -function of the motive degenerates and becomes modular. We exemplify our general construction with these two cases in turn.

$$\underline{X_3^6(\psi_1, \psi_2) \in \mathbb{P}_{(2,1,1,1,1)}}$$

This two-parameter family extends the Brieskorn-Pham variety $X_3^6(0)$ considered in [2], where it was shown that the L -series of the Ω -motive

$$M_\Omega(X_3^6(0)) : (1, 1, 1, 1, 1) \oplus (2, 5, 5, 5, 5), \quad (68)$$

is modular, arising from the modular form

$$f_\Omega(X_3^6(0), q) = f_{4,108A}(q) \in S_4(\Gamma_0(108)), \quad (69)$$

where $f_{4,108A}$ is identified by its expansion in Table 1.

The deformed Ω -motive of the two-parameter family $X_3^6(\psi_1, \psi_2)$ is parametrized by the orbit

$$M_\Omega(X_3^6(\psi_1, \psi_2)) : \langle M_\Omega(X_3^6(0), J^1 = (0, 0, 0, 3, 3), J^2 = (0, 3, 3, 0, 0)) \rangle, \quad (70)$$

generated by the two deformation vectors. While generically this motive is of rank eight, it degenerates for the singular fibers. For the fiber at $(\psi_1, \psi_2) = (1, 1)$, we find for the motivic L -function

$$L_\Omega(X_3^6(1, 1), s) = 1 + \frac{20}{7^s} - \frac{70}{13^s} + \frac{56}{19^s} + \frac{308}{31^s} + \frac{110}{37^s} + \dots \quad (71)$$

Here the bad primes are $p = 2, 3$. This L -series has complex multiplication and is determined by the Hecke L -series of a Hecke Größencharacter ψ_H associated to the Eisenstein field $\mathbb{Q}(\sqrt{-3})$

$$L_\Omega(X_3^6(1, 1), s) = L(\psi_H^3, s). \quad (72)$$

Hecke's theory therefore implies that this L -series is modular. The associated modular form admits a closed expression in terms of the Dedekind eta function

$$f_\Omega(X_3^6(1, 1), q) = \eta(q^3)^8 \in S_4(\Gamma_0(9)), \quad (73)$$

denoted by $f_{4,9}(q)$ in Table 1. This CM form is furthermore string theoretic because the Hecke indefinite modular form at the conformal level $k = 1$ is given by the theta function

$\Theta_{1,1}^1(q) = \eta^2(q)$. More details about the structure of the Hecke indefinite modular forms $\Theta_{\ell,m}^k(q)$ associated to the affine Kac-Moody algebra of $SU(2)$ can be found in [35]. This modular form has been discussed previously in a string theoretic context in the analysis of a mirror pair involving a rigid Calabi-Yau manifold [3].

$$\underline{X_3^8(\psi_1, \psi_2) \subset \mathbb{P}_{(4,1,1,1,1)}}$$

We consider the singular fiber with $(\psi_1, \psi_2) = (1, 1)$ of the octic family $X_3^8(\psi_1, \psi_2)$. Using the Gauss products $\mathbb{G}_p^n(u)$ defined above, the Ω -motive for the Brieskorn-Pham fiber $X_3^8(0)$ can be represented as (17) for $8|(p-1)$ with

$$M_\Omega(X_3^8(0)) : (1, 1, 1, 1, 1) \oplus (1, 3, 3, 3, 3) \oplus (1, 5, 5, 5, 5) \oplus (1, 7, 7, 7, 7). \quad (74)$$

This leads to a deformed motive $M_\Omega(X_3^8(\psi_1, \psi_2))$ of type (18) that is generated from the Brieskorn-Pham Ω -motive by the deformation vectors as

$$M_\Omega(X_3^8(\psi_1, \psi_2)) : \langle M_{\Omega(0,0)}, J^1 = (0, 4, 4, 0, 0), J^2 = (0, 0, 0, 4, 4) \rangle. \quad (75)$$

The resulting motive has rank eight, but degenerates for the $(\psi_1, \psi_2) = (1, 1)$ fiber, leading to the L -function

$$L_\Omega(X_3^8(1, 1), s) = 1 - \frac{22}{5^s} + \frac{18}{13^s} - \frac{94}{17^s} + \frac{130}{29^s} - \frac{214}{37^s} - \frac{230}{41^s} + \dots \quad (76)$$

with bad prime $p = 2$.

This L -function agrees with the twist of the L -series of a complex multiplication modular form of weight four and level $N = 32$, given by $f_{4,32} \in S_4(\Gamma_0(32))$ of Table 1. The twist character is given by χ_2 and the twisted form $f_{4,32} \otimes \chi_2$ has level $N = 64$, leading to

$$L_\Omega(X_3^8(1, 1), s) = L(f_{4,32} \otimes \chi_2, s), \quad (77)$$

where χ_2 is again a Dirichlet character.

More modularity results for CY threefolds

Other threefolds can be analyzed in a similar way. In Table 3 we summarize the results for singular fibers in a number of different weighted one-parameter and two-parameter families of the type (67) in weighted ambient spaces $\mathbb{P}_{(w_0, \dots, w_4)}$.

CY 3fold	Motivic modular form
$X_3^6(1) \subset \mathbb{P}_{(2,1,1,1,1)}$	$f_{4,108A}(q) \otimes \chi_3 \in S_4(\Gamma_0(432))$
$X_3^6(1, 1) \subset \mathbb{P}_{(2,1,1,1,1)}$	$f_{4,9} \in S_4(\Gamma_0(9))$
$X_3^8(1, 1) \subset \mathbb{P}_{(4,1,1,1,1)}$	$f_{4,32}(q) \otimes \chi_2 \in S_4(\Gamma_0(64))$
$X_3^{12A}(1) \subset \mathbb{P}_{(6,2,2,1,1)}$	$f_{4,108A}(q) \in S_4(\Gamma_0(108))$
$X_3^{12A}(1, 1) \subset \mathbb{P}_{(6,2,2,1,1)}$	$f_{4,9}(q) \otimes \chi_3 \in S_4(\Gamma_0(144))$
$X_3^{12B}(1) \subset \mathbb{P}_{(4,4,2,1,1)}$	$f_{4,108B}(q) \otimes \chi_3 \in S_4(\Gamma_0(432))$

Table 3. *Modular fibers in one- and two-parameter families of Calabi-Yau hypersurfaces.*

Remarks

1) The motives of the two-parameter threefold families of Table 3 are generically of rank eight. The L -functions of the fibers at $(\psi_1, \psi_2) = (1, 1)$ are of CM-type, hence modular by Hecke's theory (a list of weight four CM modular forms was constructed in ref. [32]).

2) The motive of the generic fiber of the one-parameter threefold family of degree twelve given by $X_3^{12B}(\psi) \subset \mathbb{P}_{(4,4,2,1,1)}$ has rank four. At the singular point $\psi = 1$ this motive degenerates and its L -series turns out to be a twist of the L -series of the rank two Ω -motive $M_{\Omega(0)}$ of the weighted Fermat hypersurface $X_3^6(0) \subset \mathbb{P}_{(2,1,1,1,1)}$. As mentioned above, the L -series $L_{\Omega}(X_3^6(0), s)$ is known to be modular [2], leading to a form of weight four and level $N = 108$. The character that produces the correct signs is given by χ_3 . This leads to the relation

$$L_{\Omega}(X_3^{12B}(1), s) = L_{\Omega}(X_3^6(0), s) \otimes \chi_3, \quad (78)$$

which via the inverse Mellin transform leads to a modular form of the same weight and level $N = 432$.

3) The last three families in Table 3 are K3 fibrations whose generic fibers are given by degree six K3 hypersurfaces in the ambient space $\mathbb{P}_{(3,1,1,1)}$ for the first two examples, and by the degree six K3 hypersurface embedded in $\mathbb{P}_{(2,2,1,1)}$ for the final entry of the table. The Brieskorn-Pham surfaces in these ambient spaces are extremal K3 surfaces, i.e. their Picard numbers are maximal. Extremal K3 surfaces are known to be modular over some number field by the work of Shioda and Inose [37], but string theoretically only modular forms with integral coefficients have been relevant so far. This motivates the question about the precise automorphy properties of the K3-submotives for the singular threefold fibers.

To apply our construction to the K3-submotives of the general fiber of the family of threefolds, consider the deformation of the Ω -motive of the Brieskorn-Pham K3 fiber. In the first one-parameter family $X_3^{12A}(\psi)$ this leads to the motive

$$M_{\text{K3}}(X_3^{12A}(\psi)) : (1, 1, 1, 2, 0) \oplus (1, 1, 1, 2, 0) \oplus (1, 5, 5, 10, 0) \oplus (1, 5, 5, 4, 6), \quad (79)$$

which has the L -function

$$L(M_{\text{K3}}(X_3^{12A}(1)), s) = L_{\Omega}(X_2^{6A}(0), s), \quad (80)$$

and is therefore modular, as noted in our K3 discussion above.

The K3 submotive of the second 1-parameter family of K3 fibrations $X_3^{12B}(\psi)$ is given by

$$M_{\text{K3}}(X_3^{12B}(\psi)) : (1, 1, 1, 2, 0) \oplus (1, 1, 1, 8, 6) \oplus (2, 2, 5, 10, 0) \oplus (2, 2, 5, 4, 6). \quad (81)$$

The L -function of this motive is again modular, taking the form

$$L(M_{\text{K3}}(X_3^{12B}(1)), s) = L_{\Omega}(X_2^{6B}(0), s) = L(\eta^3(q^2)\eta^3(q^6) \otimes \chi_3, s). \quad (82)$$

6 CFT aspects of modular singular fibers

The original motivation for the program of identifying geometric (i.e. motivic) modular forms with forms arising from the worldsheet was the idea of using these techniques to provide a string theoretic construction of spacetime [34, 2]. In this section we consider a second application that arises in the context of phase transitions in string theory. Such transitions were originally considered in string theory via the splitting and contraction procedure of ref. [12]. They provide a connection between Calabi-Yau families with different Hodge numbers via singular spaces whose degenerations involve singular points that are nodes. It was shown in [13] that the splitting-contraction construction connects the class of all complete intersection Calabi-Yau manifolds [12]. The geometry of such conifold type spaces was investigated in detail in [14, 15], and the physical implications were discussed in [10, 11]. Conifold configurations are not the only type of singularity encountered in the moduli space of Calabi-Yau varieties, and these other types of degenerations have so far received little attention.

Related to the question of a better understanding of the physical nature of more general singularities is the old problem of dealing with deformations of conformal field theories. Starting

from a rational point in a CFT moduli space, it has proven difficult to understand the behavior of the deformed theories when one moves away from the rational point along a marginal direction. It is therefore of interest to ask whether the modular forms of deformed Calabi-Yau manifolds with non-nodal singularities can be related to physical quantities. In this context an observation made above becomes relevant.

In Section 4 we noted that some motivic modular forms identified for singular fibers of deformed weighted Fermat hypersurfaces are identical to modular forms derived from Brieskorn-Pham varieties, i.e. for smooth fibers at $\psi_J = 0$. This immediately implies that the motivic geometry of such fibers is determined by a rational conformal field theory, even though it is obtained from a deformed theory and therefore might have been expected to be a complicated object in an irrational theory that is difficult to identify. In our first example, the K3 surface family $X_2^{6B}(\psi) \subset \mathbb{P}_{(1,1,2,2)}$, the motivic L -series at $\psi = 1$ was shown to be identical to the Ω -motivic L -series of the Brieskorn-Pham fiber $X_2^{6A}(0) \subset \mathbb{P}_{(1,1,1,3)}$. The modular form $f_\Omega(X_2^{6B}(0), q) \in S_3(\Gamma_1(27))$ was determined in [1] as $f_\Omega(X_2^{6A}(0), q) = \eta^2(q^3)\eta^2(q^9)\vartheta(q^3)$. This result is surprising for several reasons. First, as noted above, the L -series of this form can also be viewed as a Hecke L -series $L_\Omega(X_2^{6A}(0), s) = L(\Psi_H^2, s)$ for the Hecke character $\Psi_H = \Psi_{6A}$ defined previously. This shows that this L -series is the symmetric square of the L -series defined by the Mellin transform of a weight two modular form $f_{2,27}(q) \in S_2(\Gamma_0(27))$. This modular form of weight two turns out to be the factor of the weight three form determined by the Dedekind eta function

$$f_{2,27}(q) = \eta^2(q^3)\eta^2(q^9). \quad (83)$$

The origin of the form $f_{2,27}(q)$ is natural from two quite different viewpoints. First, it is precisely the modular form of the cubic Fermat elliptic curve E^3 of eq. (57) which appears as a fiber of the elliptic family $E^3(\psi)$ defined by the elliptically fibered K3 family $X_2^{6B}(\psi)$. From this geometric perspective, the appearance of the string theoretic elliptic modular form $f_{2,27}(q)$ is expected. Second, the Brieskorn-Pham fiber in the family $X_2^{6B}(\psi)$ has as its corresponding Gepner model the GSO projected tensor product $(k = 1)^2 \otimes (k = 4)^2$ of minimal $N = 2$ supersymmetric conformal field theories, where k denotes the affine level of the model with central charge $c = 3k/(k + 2)$. Each factor leads to Hecke indefinite modular forms

$$\Theta_{\ell,m}^k(\tau) = \sum_{\substack{-|x| < y \leq |x| \\ (x,y) \text{ or } (\frac{1}{2}-x, \frac{1}{2}+y) \\ \in \mathbf{Z}^2 + (\frac{\ell+1}{2(k+2)}, \frac{m}{2k})}} \text{sign}(x)e^{2\pi i\tau((k+2)x^2 - ky^2)}, \quad (84)$$

which are related to the parafermionic Kac-Peterson string functions $c_{\ell,m}^k(\tau)$ as

$$\Theta_{\ell,m}^k(\tau) = \eta^3(\tau)c_{\ell,m}^k(\tau). \quad (85)$$

These string functions, in combination with standard theta functions, completely determine the characters of the $N = 2$ superconformal model [4]. For affine level $k = 1$, there exists a unique Hecke indefinite modular form, given by

$$\Theta_{1,1}^1(q) = \eta^2(q). \quad (86)$$

This then leads to the worldsheet interpretation of the motive of the fiber $X_2^{6B}(1)$ by recognizing that the weight two building block can be written as $f_{2,27}(q) = \Theta_{1,1}^1(q^3)\Theta_{1,1}^1(q^9)$.

Other examples can be analyzed in a similar way, and in Table 4 we summarize those results that explicate the emergence of modular forms that arise from deformed Ω -motives in singular configurations, but which also appear as motivic forms in Calabi-Yau varieties of weighted Fermat type. The motivic L -functions of all the fibers $X(\psi)$ in Table 4 are either identical to motivic L -functions of Brieskorn-Pham hypersurfaces $X'(0)$ or twists of such L -series that we have previously shown to be modular in terms of string theoretic modular forms [1, 2]. Thus we have

$$L_{\Omega}(X(\psi), s) = L(f_{\Omega}(X'(0)) \otimes \chi, s), \quad (87)$$

where χ is a (possibly trivial) Dirichlet character, and $f_{\Omega}(X'(0), q)$ is a known modular form $f_{\Omega}(X'(0), q) \in S_w(\Gamma_0(N))$, where $w = n + 1$ and N is the level. These modular forms are all of CM-type, hence modularity follows directly from Hecke's theory. The string theoretic interpretation of these forms proceeds again in terms of the Hecke indefinite modular forms defined above.

CY fiber $X_n^d(1)$	BP variety $X_n^d(0)$	Modular form
$X_2^{6B}(1) \subset \mathbb{P}_{(2,2,1,1)}$	$X_2^{6A}(0) \subset \mathbb{P}_{(3,1,1,1)}$	$f_{3,27}$
$X_2^8(1) \subset \mathbb{P}_{(4,2,1,1)}$	$X_2^4(0) \subset \mathbb{P}_3$	$f_{3,16} \otimes \chi_2$
$X_2^{12A}(1) \subset \mathbb{P}_{(6,4,1,1)}$	$X_2^{6A}(0) \subset \mathbb{P}_{(3,1,1,1)}$	$f_{3,27}(q) \otimes \chi_3$
$X_3^{12B}(1) \subset \mathbb{P}_{(4,4,2,1,1)}$	$X_3^6(0) \subset \mathbb{P}_{(2,1,1,1,1)}$	$f_{4,108B}(q) \otimes \chi_3$

Table 4. Relations between modular forms of singular fibers and forms of Brieskorn-Pham fibers.

The results described for the examples above, and their extension to the varieties in Table 4, show that the structure of these singular fibers is determined by the modular forms that emerge at the smooth Brieskorn-Pham point of the family corresponding to the Gepner point. The existence of such L -correlated pairs suggests that the singular phases that occur in the families $X_n^d(\psi_1, \psi_2)$ considered here admit a physical interpretation and that the conformal field theory is much better under control than previously thought.

7 Motivic dimensional transmutation

In the previous section we have shown that singular Calabi-Yau varieties with high Milnor numbers may lead to geometric modular forms that can be identified with string theoretic forms on the worldsheet. In the examples discussed so far, the degeneration of the motive only affects the rank of the motive, not its weight. The weight of the motive of a smooth variety is a characteristic that is determined by the degree of the cohomology associated with the motive. In the case of the Ω -motive of smooth Calabi-Yau varieties this is the intermediate cohomology, in which case the weight $\text{wt}(M_\Omega)$ is given by $\text{wt}(M_\Omega) = n$, where n is the complex dimension of the variety. For the special case of smooth Calabi-Yau varieties the weight of the Ω -motive can therefore be viewed as a proxy for the dimension of the space. The weight of the resulting modular form $w(f)$ is then given by $w(f) = \text{wt}(M_\Omega) + 1$. In this section we will discuss examples in which the degeneration of the motive induced by the singularities reduces not only the rank but also the weight, leading to a kind of dimensional transmutation of the motive at the singular fibers of high Milnor number.

Perhaps the simplest example in which the phenomenon of motivic dimensional transmutation happens is the quartic K3 surface of type (4) with $\psi_1 = 0$, $\psi_2 = \psi$, and $b = 2$. In this case the deformed motive

$$M_\Omega(X_2^4(\psi)) : \quad (1, 1, 1, 1) \oplus (1, 1, 3, 3) \oplus (3, 3, 3, 3) \oplus (3, 3, 1, 1) \quad (88)$$

for the generic fiber is of rank four, the same rank as for the motives of the generic fibers of all the one-parameter K3 families considered in the previous section. The L -function of this motive is given by

$$L_\Omega(X_2^4(1), s) = 1 + \frac{2}{5^s} - \frac{6}{13^s} + \frac{2}{17^s} + \frac{10}{29^s} + \frac{2}{37^s} + \dots \quad (89)$$

This shows that this Ω -motivic L -function of a complex surface is in fact identical to the L -series of the quartic weighted Fermat elliptic curve $E^4 \subset \mathbb{P}_{(2,1,1)}$

$$L_{\Omega}(X_2^4(1), s) = L(E^4, s). \quad (90)$$

The modular form of the curve E^4 was determined in [36] to be given by the weight two form at level 64

$$L(E^4, s) = L(\eta^2(q^4)\eta^2(q^8) \otimes \chi_2, s), \quad (91)$$

where the expansion of $f_{2,32}(q) = \eta^2(q^4)\eta^2(q^8) \in S_2(\Gamma_0(32))$ is given in Table 1. It follows that this L -series is string theoretic at conformal level $k = 2$, since the modular forms $f_{2,32}$ can be written in terms of the Hecke indefinite modular forms as $\Theta_{1,1}^2(q) = \eta^2(q)\eta^2(q^2)$. This leads to the string theoretic identification of the motivic modular form (90), and therefore of the highly degenerate motive coming from the singular fiber of the quartic K3 family $X_2^4(\psi)$.

For Calabi-Yau threefolds similar phenomena emerge. Consider, for example, the two-parameter family of threefolds $X_3^{8B}(\psi_1, \psi_2) \subset \mathbb{P}_{(2,2,2,1,1)}$ of type (4) with $a = 2$ and $b = 4$. The deformed Ω -motive for the general fiber is of rank eight

$$M_{\Omega}(X(\psi_1, \psi_2)) : \quad \langle M_{\Omega(0)}, J^1 = (0, 2, 2, 0, 0), J^2 = (0, 0, 0, 4, 4) \rangle, \quad (92)$$

the same rank as for all deformed motives of the generic fibers of the two-parameter families discussed in the previous section. The L -function of this motive is identical to the motivic L -function of the octic K3 surface $X_2^8(1) \subset \mathbb{P}_{(4,2,1,1)}$

$$L_{\Omega}(X_3^{8B}(1, 1), s) = L_{\Omega}(X_2^8(1), s), \quad (93)$$

considered earlier in this paper.

We have seen in Section 5 that the L -series of $X_2^8(1)$ is modular, given in fact by the χ_2 -twist of the L -function of the Fermat K3 surface $X_2^4(0)$. The resulting CM modular form is therefore string modular according to the results in [1]. A second Calabi-Yau threefold fiber of this type appears in the twoparameter family $X_3^{12C}(\psi_1, \psi_2) \subset \mathbb{P}_{(4,3,3,1,1)}$, where we set $(\psi_1, \psi_2) = (1, 1)$. Again, the L -series is identical to that of a K3 surface, in this case $X_2^{12}(1) \subset \mathbb{P}_{(6,4,1,1)}$, which in turn is the twist of the L -series of the degree six weighted Fermat K3 hypersurface $X_2^{6A}(0) \subset \mathbb{P}_{(3,1,1,1)}$. The Ω -motive of this latter surface is known [1] to be modular with $L_{\Omega}(X_2^{6A}(0), s) = L(f_{3,27}, s)$. We collect the modular data of these examples in Table 5.

Fiber	Modular form	BP variety	Modular form
$X_2^4(1) \subset \mathbb{P}_3$	$f_{2,32} \otimes \chi_2$	$E^4 \subset \mathbb{P}_{(2,1,1)}$	$f_{2,32} \otimes \chi_2$
$X_3^{8A}(1, 1) \subset \mathbb{P}_{(2,2,2,1,1)}$	$f_{3,16} \otimes \chi_2$	$X_2^4(0) \subset \mathbb{P}_3$	$f_{3,16}$
$X_3^{12C}(1, 1) \subset \mathbb{P}_{(4,3,3,1,1)}$	$f_{3,27} \otimes \chi_3$	$X_2^{6A}(0) \subset \mathbb{P}_{(3,1,1,1)}$	$f_{3,27}$

Table. 5. *Examples with modular weight reduction, leading to forms $f_{w,N}$ with $w < n + 1$.*

The results discussed here and in the previous section show that both the effective rank and the modular weight of a motive can change dramatically as the family varies over the moduli space. In the previous section we saw examples in which the effective rank of the motive changed such that a motive that was expected to be automorphic of high degree reduced to something modular with weight precisely matching that expected from the dimension of the variety (or rather the level of the intermediate cohomology). In contrast, in this section we encountered examples in which not only was the effective rank reduced, but also the modular weight.

The results discussed here not only show that a family of high rank motives can become modular at singular points in the moduli space, but that the resulting modular forms can arise from lower-dimensional varieties. These modular forms retain their string theoretic nature even in such highly degenerate situations. The phenomenon of dimensional transmutation of motives of singular Calabi-Yau spaces thus suggests that such singular points in the moduli space should define transitions between string vacua of different dimensions. It might therefore be possible to embed into string theory considerations of earlier papers that concerned different decompactification mechanisms in lower-dimensional ad-hoc models [38, 39, 40].

8 Concluding remarks

In this paper we have taken the first steps in generalizing the higher-dimensional string modularity results obtained in [1, 2, 3] for Brieskorn-Pham type varieties to families of Calabi-Yau spaces. This establishes that the program of constructing an emergent spacetime within string theory via modular methods transcends the framework of exactly solvable rational conformal field theories and generalizes to deformations of such theories via marginal operators.

An unexpected result which indicates the usefulness of the modular techniques is that the

string theoretic interpretation shows that completely different fibers in the moduli space of K3 surfaces lead to identical modular forms. Since these modular forms determine the structure of the underlying conformal field theory, this shows that the worldsheet theories lead to identical modular structures, although the geometry is rather different. This implies that deformations of rational conformal field theories inherit at least part of the structure of the rational points in moduli space, in particular as far as the structure of spacetime is concerned.

Furthermore, rather surprisingly the modular forms associated to the deformed Ω -motives of singular fibers can be identical to modular forms arising from the Ω -motives of smooth Brieskorn-Pham manifolds. The construction of these forms in terms of the forms of the worldsheet suggests that string theory might be viable on spaces with singularities of high multiplicity and that such configurations may be transit points of phase transition, similar to conifolds in the case of node-type singularities.

Finally, we have observed the phenomenon that at certain singular fibers with higher multiplicity singular points the L -function can degenerate so that the motive reduces not only its rank but also its weight, leading to what might be called motivic dimensional transmutation. Even in these cases, the modular forms that appear in our examples admit a string theoretic interpretation.

It would be of interest to see whether the conformal field theoretic techniques developed by Wendland [16] over the past years can be used to illuminate the CFT structure of the singular fibers described in this paper.

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References

- [1] R. Schimmrigk, *The Langlands program and string modular K3 surfaces*, Nucl. Phys. **B771** (2007) 143 – 166, arXiv: hep-th/0603234
- [2] R. Schimmrigk, *Emergent spacetime from modular motives*, Commun. Math. Phys., in press, arXiv: 0812.4450 [hep-th]
- [3] S. Kharel, M. Lynker and R. Schimmrigk, *String modular motives for mirrors of rigid Calabi-Yau varieties*, in MODULAR FORMS AND STRING DUALITY, ed. N. Yui, Fields Inst. Commun. **54** (2008) pp. 47 – 63, arXiv: 0908.1256 [math.AG]
- [4] D. Gepner, *Spacetime supersymmetry in compactified string theory and superconformal models*, Nucl. Phys. **B296** (1988) 757 – 778
- [5] E.J. Martinec, *Algebraic geometry and effective Lagrangians*, Phys. Lett. **B217** (1989) 431
- [6] C. Vafa and N.P. Warner, *Catastrophes and the classification of conformal theories*, Phys. Lett. **B218** (1989) 51
- [7] W. Lerche, C. Vafa and N.P. Warner, *Chiral rings in $N = 2$ superconformal theories*, Nucl. Phys. **B324** (1989) 427
- [8] E. Witten, *Phases of $N = 2$ theories in two dimensions*, Nucl. Phys. **B403** (1993) 159 – 222, arXiv: hep-th/9301042
- [9] Y. Kazama and H. Suzuki, *New $N = 2$ superconformal field theories and superstring compactification*, Nucl. Phys. **B321** (1989) 232
- [10] A. Strominger, *Massless black holes and conifolds in string theory*, Nucl. Phys. **B451** (1995) 96 – 108
- [11] B.R. Greene, D. Morrison and A. Strominger, *Black hole condensation and the unification of string vacua*, Nucl. Phys. **B451** (1995) 109 – 120, arXiv: hep-th/9504145
- [12] P. Candelas, A. Dale, C.A. Lütken and R. Schimmrigk, *Complete intersection Calabi-Yau manifolds*, Nucl. Phys. **B298** (1988) 493 – 525
- [13] P. Green and T. Hübsch, *Connecting moduli spaces of Calabi-Yau threefolds*, Commun. Math. Phys. **119** (1988) 431 – 441; *Phase transitions among (many of) Calabi-Yau compactifications*, Phys. Rev. Lett. **61** (1988) 1163

- [14] P. Candelas, P. Green and T. Hübsch, *Finite distance between distinct Calabi-Yau manifolds*, Phys. Rev. Lett. **62** (1989) 1956 – 1959; *Rolling among Calabi-Yau vacua*, Nucl. Phys. **B330** (1990) 49 – 102
- [15] P. Candelas and X. de la Ossa, *Comments on conifolds*, Nucl. Phys. **B342** (1990) 246 – 268
- [16] K. Wendland, *A family of SCFTs hosting all "very attractive" relatives of the $(2)^4$ Gepner model*, JHEP **0603** (2006) 102, arXiv: hep-th/0512223
- [17] A. Weil, *Number of solutions of equations in finite fields*, Bull. Am. Math. Soc. **55** (1949) 497 – 508
- [18] A. Grothendieck, *Formulé de Lefschetz et rationalité de fonction de L*, Séminaire Bourbaki **279**, 1964/1965, 1 – 15
- [19] P. Deligne, *La conjecture de Weil I*, Publ. Math. IHES **43** (1974) 273 – 307
- [20] P. Deligne, *La conjecture de Weil II*, Publ. Math. IHES **52** (1980) 137 – 252
- [21] U. Jannsen, S. Kleiman and J.-P. Serre, *Motives*, AMS 1994
- [22] R. Schimmrigk, *Critical string vacua from noncritical manifolds*, Phys. Rev. Lett. **70** (1993) 3688 – 3691, arXiv: hep-th/9211001
- [23] P. Candelas, E. Derrick and L. Parkes, *Generalized Calabi-Yau manifolds and the mirror of a rigid manifold*, Nucl. Phys. **B407** (1993) 115 – 154, arXiv: hep-th/9304045
- [24] R. Schimmrigk, *Mirror symmetry and string vacua from a special class of Fano varieties*, Int. J. Mod. Phys. **A11** (1996) 3049 – 3096, arXiv: hep-th/9405086
- [25] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, *Calabi-Yau manifolds over finite fields*, I, preprint, arXiv: hep-th/0012233
- [26] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, *Calabi-Yau manifolds over finite fields*, II, Fields Inst. Commun. **38** (2003) 121 – 157, arXiv: hep-th/0402133
- [27] P. Candelas and X. de la Ossa, *The zeta-function of a p -adic manifolds, Dwork theory for physicists*, arXiv: 0705.2056 [hep-th]
- [28] S. Kadir, *The arithmetic of Calabi-Yau manifolds and mirror symmetry*, Ph.D. thesis, 2004, arXiv: hep-th/0409202

- [29] S. Kadir, *Arithmetic mirror symmetry for a two-parameter family of Calabi-Yau manifolds*, in MIRROR SYMMETRY V, eds. N. Yui, S.-T. Yau and J. Lewis, AMS/IP, 2006
- [30] N. Koblitz, *Points on certain families of hypersurfaces over finite fields*, Compos. Math. **48** (1983) 3 – 23
- [31] C. Meyer, *Newforms of weight four for $\Gamma_0(N)$ with rational coefficients*, preprint, 2005
- [32] M. Schütt, *Hecke eigenforms and the arithmetic of singular K3 surfaces*, dissertation 2006
- [33] E. Hecke, *Mathematische Werke*, 1959
- [34] R. Schimmrigk and S. Underwood, *The Shimura-Taniyama conjecture and conformal field theory*, J. Geom. Phys. **48** (2003) 169 – 189, arXiv: hep-th/0211284
- [35] V.G. Kac and D.H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. Math. **53** (1984) 125 – 264
- [36] M. Lynker and R. Schimmrigk, *Geometric Kac-Moody modularity*, J. Geom. Phys. **56** (2006) 843 – 863, arXiv: hep-th/0410189
- [37] T. Shioda and H. Inose, *On singular K3 surfaces*, in COMPLEX ANALYSIS AND ALGEBRAIC GEOMETRY, Cambridge University Press, 1977, pp. 119 – 136
- [38] S.B. Giddings and R.C. Myers, *Spontaneous decompactification*, Phys. Rev. **D70** (2004) 046005, arXiv: hep-th/0404220
- [39] S.M. Carroll, M.C. Johnson and L. Randall, *Dynamical compactification from de Sitter space*, arXiv: 0904.3115 [hep-th]
- [40] J.J. Blanco-Pillado, D. Schwartz-Perlov and A. Vilenkin, *Transdimensional tunneling in the multiverse*, arXiv: 09124082 [hep-th]